PR 502: Robot Dynamics & Control

Robot Kinematics: Articulated Robots

Asanga Ratnaweera
Department of Mechanical Engineering
Articulated Robots

- **Home position**
  - When all the joints are in zero position, the manipulator is said to be in *home position.*

Denavit-Hartenberg (DH) Representation

- Developed by Denavit and Hartenberg in 1955 for kinematic modeling of lower pairs.
- Has become a standard way of representing robots and modeling their motions.
- However, direct modeling techniques learned before are faster and straight forward.
- Quite useful for Articulated robot modeling.
Links, Joints and Their Parameters

- Mechanical manipulator consists of sequence of rigid bodies (links) connected either revolute or prismatic joint.
- Each joint-link pair constitutes one degree of freedom (dof).
- Hence, for n dof system has n number of links.
- Usually the first link (link 0) is attached to a supporting base and last link is attached with the tool.

DH Coordinate frames

- Establishing base coordinate system:
  - A right handed Cartesian coordinate system XYZ or $x_0, y_0, z_0$ is assigned to the base of the manipulator with the $z_0$ axis lying along the axis of motion for the 1st link (joint 1) and pointing towards the shoulder of the robot.
Establishing the joint axis:

- All joints without exception are represented by a Z axis.
- If the joint is revolute the Z axis is the axis of rotation.
- If the joint is prismatic, Z axis is along the direction of the linear motion.
DH Coordinate frames

- Establishing the joint axis:
  - Defining X axis
    - Assign the X axis along the common normal between two Z axes.
    - If two z axes are parallel then assign X axis along the common normal to the previous joint.
    - If two Z axes are intersecting each other, assign the x axis along a line perpendicular to the plane formed by the two Z axes.

Common normal between two z axes of joint 1 and 2
DH Coordinate frames

Ex: Assign coordinate frames
DH Coordinate frames

Denavit-Hartenberg Parameters
Coordinate frames for DH Parameter

- Four parameters \((a,a,d,\theta)\) are associated with each link of a manipulator.
- \(Z_i\) axis is aligned with axis \(i\), its direction being arbitrary. The choice of direction defines the positive sense of joint variable \(\theta_i\).
- The \(X_i\) axis is perpendicular to axis \(Z_{i-1}\) and axis \(Z_i\) and point away from axis \(Z_{i+1}\). Therefore, \(X_i\) is along the common normal \(CD\).
- The origin of the \(i^{th}\) coordinate frame, is located at the intersection of axis of joint \((i+1)\), that is axis \(i\) and the common normal between axes \((i-1)\) and \(i\).
- \(Y_i\) completes the right and orthogonal coordinate frame \(i\).

DH Parameter

\(\alpha\): Amount of rotation around the common perpendicular \((X axis)\) so that the joint axes are parallel.

- Ex: \(\alpha_i\) is how much you have to rotate \(Z(i-1)\) about \(X_i\) axis so that the \(Z(i-1)\) is pointing in the same direction as the \(Z(i)\) axis.
- Positive rotation follows the right hand rule.
DH Coordinate frames

- Parameter $\alpha$

DH Parameter

- $a$: the perpendicular distance between the adjacent joint axes (Z axes). i.e. distance between $Z_{(i-1)}$ and $Z_{(i)}$ along $X_i$ axis
  - ex: $a_i$ is the perpendicular distance between $Z_{(i-1)}$ and $Z_{(i)}$ axes.
DH Parameters

- Parameter $a$

DH Parameters: $\alpha$ and $a$

- If Z – axes (extended if required) are
  - Collinear lines: $\alpha = 0$ and $a = 0$
  - Parallel lines: $\alpha = 0$ and $a \neq 0$
  - Intersecting lines: $\alpha \neq 0$ and $a = 0$
  - Skew lines: $\alpha \neq 0$ and $a \neq 0$
DH Parameters

- **d_i**: The displacement along the Z_{i-1} axis needed to intersect the a_i common perpendicular and Z_i axis at the origin of the i-1^th frame.
  - In other words, displacement along the Z_{i-1} to intersect the X_i and Z_{i-1} at its origin.

DH Parameters

- **θ_i**: rotation of X_{i-1} about Z_{i-1} axis needed to align the a_i common perpendicular to the a_{i-1} common perpendicular. (Usually the joint rotation from the home position)
  - In other words, rotation about Z_{i-1} to align the X_{i-1} and X_i axes.
  - Positive rotation follows the right hand rule.
  - Note: a, α are called link parameters and d, θ are called joint parameters
Denavit-Hartenberg Parameters

- $X_i$ is along the common normal of $Z_i$ and $Z_{(i-1)}$ hence $a_i$ measured along $X_i$
- $d_i$ measured along $Z_{i-1}$
- $\theta_i$ (rotation of $X_{(i-1)}$) about $Z_{i-1}$
- $\alpha_i$ (rotation of $Z_{(i-1)}$) about $X_i$
- All angle measurements should be based on the right hand rule (counterclockwise positive)
- Note: the frame $i-1$ is assigned to joint $i$ and so forth

Denavit-Hartenberg Transformation

- Therefore, the transformation is in the reverse order:

$$i^{-1}T = T_z(\theta_i)T_z(d_i)T_x(a_i)T_x(\alpha_i)$$

- Thus, the DH transformation matrix:

$$\begin{bmatrix}
\cos \theta_i & -\sin \theta_i & \sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \theta_i & \cos \theta_i \cos \alpha_i & -\cos \theta_i \sin \alpha_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & d_i & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}$$
Manipulator transformation matrix

- The position and orientation of the tool frame relative to the base frame can be found by considering \( n \) consecutive link transformation matrices relating frames fixed to adjacent links.

\[
T = ^0T_n = ^0T_1^T_2 ................ \ T_{n-1}^T_n
\]

- \(^{i-1}T_i\) for \(i=1,2,...,n\) is Homogeneous link transformation matrix between frame \((i-1)\) and \(i\)

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Manipulator transformation matrix

- The kinematic model provides the functional relationship between the tool frame (end effector) position and orientation and displacement of each link \((q_i)\).

\[
T = f(q_i) = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

- For the known joint displacements \(q_i\) \((i=1,2,...,n)\) the end effector orientation \((n,o,a)\) and position \(p\) can be computed from the above equation.
Example: 1

Home position of the robot

DH parameter table

<table>
<thead>
<tr>
<th>Link</th>
<th>( a_i )</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
<th>( \theta_i )</th>
<th>( C\theta_i )</th>
<th>( S\theta_i )</th>
<th>( C\alpha_i )</th>
<th>( S\alpha_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0</td>
<td></td>
<td>( \theta_1 )</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

DH transformation Matrices

\[ ^0T_1(\theta_1) = \begin{bmatrix}
    C_1 & 0 & S_1 & 0 \\
    S_1 & 0 & -C_1 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} \]

\[ T_2(d_2) = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & d_2 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} \]
Example: 1

Therefore, overall transformation matrix

\[
{^0T_2} = {^0T_1}{^1T_2} = \begin{bmatrix}
C_1 & 0 & 0 & d_2 S_1 \\
S_1 & 0 & -C_1 & -d_2 C_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Comparing with the general form of the homogeneous matrix

\[
\begin{bmatrix}
n_x & a_x & a_y & a_z & d_y \\
n_y & a_x & a_y & a_z & d_y \\
n_z & a_x & a_y & a_z & d_y \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
C_1 & 0 & 0 & d_2 S_1 \\
S_1 & 0 & -C_1 & -d_2 C_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Example: 1

Therefore, the coefficients

\[\begin{align*}
n_x &= C_1 \\
n_y &= S_1 \\
n_z &= 0 \\
o_x &= 0 \\
o_y &= 0 \\
o_z &= 1
\end{align*}\]

\[\begin{align*}
a_x &= S_1 \\
a_y &= -C_1 \\
a_z &= 0 \\
d_x &= d_2 S_1 \\
d_y &= -d_2 C_1 \\
d_z &= 0
\end{align*}\]

If \(\theta_1 = 120\) and \(d_2 = 200\)

\[
T_E = \begin{bmatrix}
-0.5 & 0 & 0.866 & 173.2 \\
0.866 & 0 & 0.5 & 100.0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Example: 2
RPP Manipulator

Joint axis 1
Joint axis 2
Joint axis 3
Face plate for attaching wrist

Example: 3

Joint 1 (Waist)
Joint 2 (Shoulder)
Joint 3 (Elbow)
Face plate for attaching wrist
Arm point

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Asanga Ratnaweera, Department of Mechanical Engineering
Ex: PUMA Robot