Chapter 4
Vibrations

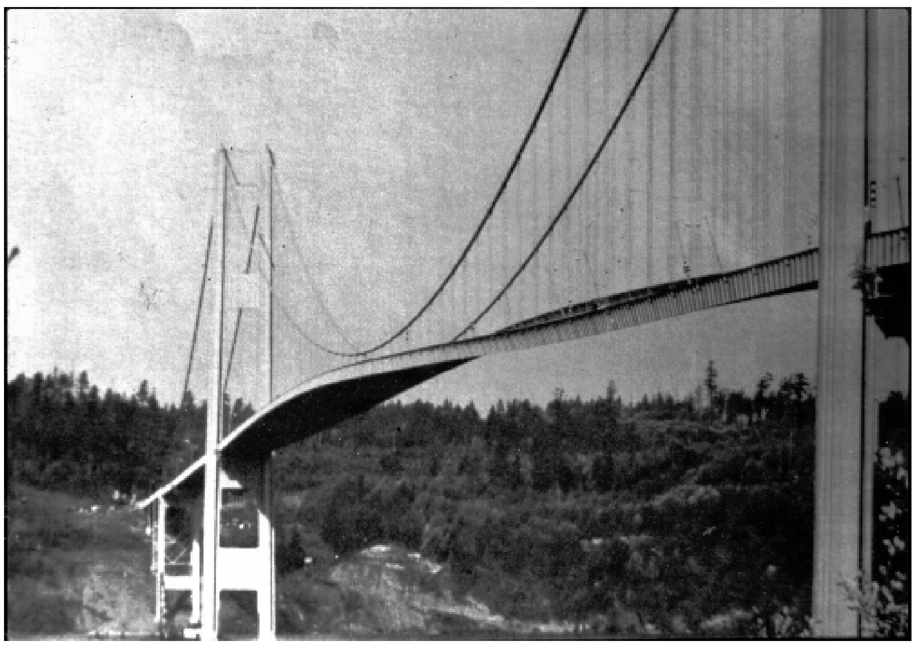


Fig. 4.1 Tacoma Narrows bridge minutes before collapse. Figure curtesy of Wikipedia.

4.1 Vibration Analysis of a Spring Mass Damper System

A first approximation of a simple vibrating system would typically result in a spring mass damper system such as the one shown in figure-4.2. For small deflections and velocities, the viscous damping force exerted by the damper can be approximated by $f_d = -C\dot{x}$ and the spring force can be approximated by $f_d = -Kx$. Considering the free body diagram of the systems shown in figure-4.3 and applying Newton's equations for the mass M yields the following linear ODE.

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t). \tag{4.1}$$

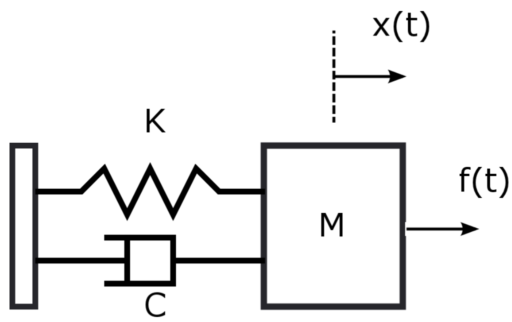


Fig. 4.2 Spring Mass Damper System.

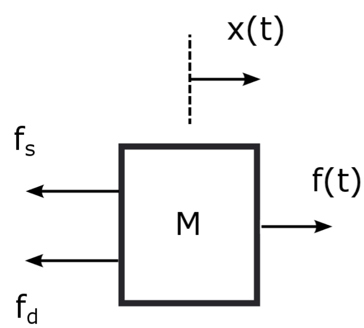


Fig. 4.3 The Free Body Diagram of the Spring Mass Damper System.

Dividing by M and setting $\omega_n^2 = \frac{K}{M}$ and $2\zeta\omega_n = \frac{C}{M}$ in (4.1) we have

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{1}{M} f(t). \tag{4.2}$$

The parameter ω_n is called the *undamped natural frequency* of the system and ζ is called the *damping ratio* of the system. The reason for this nomenclature will be apparent when we investigate the solutions of this differential equation.

Being a linear second order ODE one can solve (5.11). Since this differential equation was derived by considering small deflections and velocities of the mass it is interesting to find out if the solutions of the differential equation correspond to the physical behaviour of the SMD system for small deflections.

One way of finding the solutions of this differential equation is the Laplace transform method. For a given $u(t) = \frac{1}{M} f(t)$ that does not grow faster than an exponential function it can be shown, owing to the linearity of the system, that the solution will also not grow faster than an exponential function. Thus taking Laplace transform of both sides of (5.11) and using the linearity property of the Laplace transform we have

$$\mathscr{L}\left\{\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t)\right\} = \mathscr{L}\left\{u(t)\right\}, \tag{4.3}$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) - \dot{x}(0) - (s + 2\zeta\omega_n)x(0) = U(s). \tag{4.4}$$

This yields

$$X(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\dot{x}(0) + \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}x(0) + \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}U(s). \tag{4.5}$$

Since the Laplace is a one-to-one and onto operator its inverse exists and thus the the solution can be uniquely determined to be $x(t) = \mathscr{L}^{-1}\{X(s)\}$. Using the linearity property of the inverse we thus have

$$x(t) = \dot{x}(0)\mathscr{L}^{-1}\left\{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} + x(0)\mathscr{L}^{-1}\left\{\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}U(s)\right\}. \tag{4.6}$$

Observe that the first two terms depend on the initial conditions and do not depend on the forcing while the last term does depend on the forcing but is in depend of the initial conditions. Therefore we may breakup the solution into two parts such that $x(t) = x_{IC}(t) + x_f(t)$ where

$$x_{IC}(t) = \dot{x}(0)\mathscr{L}^{-1}\left\{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} + x(0)\mathscr{L}^{-1}\left\{\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\right\}, \tag{4.7}$$

$$x_f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}U(s)\right\}. \tag{4.8}$$

We will call the first part two parts given by $x_{IC}(t)$ that depends only on the initial conditions the *initial condition response* while we will call the last part that depends only on the forcing the *forced response*.

The inverse Laplace transforms of each of the terms are obtained by expanding the terms in a partial fraction expansion. This depends on the roots of the polynomial

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2. \tag{4.9}$$

The two roots , $-\lambda_1$ and $-\lambda_2$, of the polynomial are given by

$$-\lambda_1 = \omega_n(-\zeta + \sqrt{\zeta^2-1}),$$
$$-\lambda_2 = \omega_n(-\zeta - \sqrt{\zeta^2-1}).$$

Since the character of the solution is determined by the roots of the above polynomial it is referred to as the *characteristic polynomial* of the system. These roots will be complex and conjugate, real and distinct, or real and repeated depending on if $0 \leq \zeta < 1$, $\zeta > 1$, or $\zeta = 1$ respectively.

Let us consider the case $0 \leq \zeta < 1$. In this case the roots of the characteristic polynomial $\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ of the system are given by $-\lambda_1 = -\zeta\omega_n + i\omega_d$ and $-\lambda_2 = -\zeta\omega_n - i\omega_d$. A typical location of the roots of the characteristic polynomial in this case is shown in figure-4.4. The plot of $x_{IC}(t)$ for several different values of ζ in the range $0 \leq \zeta \leq 1$ is shown in figure-4.5.

Assume that the forcing is such that

$$\mathscr{L}\{u(t)\} = U(s) = \frac{N(s)}{(s+\beta_1)(s+\beta_2)\cdots(s+\beta_k)}.$$

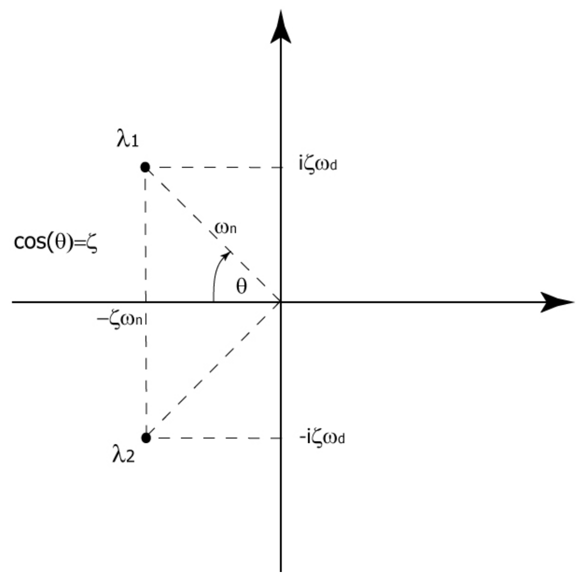


Fig. 4.4 A typical location of the roots, $\lambda = -\lambda_1, \lambda = -\lambda_2$, of the characteristic polynomial $\Delta(s)$ when $0 \leq \zeta < 1$.

When $0 \leq \zeta \leq 1$ it is a straight forward exercise in partial fraction expansion to show that (4.7) and (4.8) reduce to

$$x_{IC}(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0)}{\omega_d}\cos(\omega_d t - \pi/2) + \frac{x(0)}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi_{IC})\right)1(t), \quad (4.10)$$

$$x_f(t) = \left(|\alpha|e^{-\zeta\omega_n t}\cos(\omega_d t + \phi) + \left(r_1 e^{-\beta_1 t} + \cdots + r_k e^{-\beta_k t}\right)\right)\, 1(t), \quad (4.11)$$

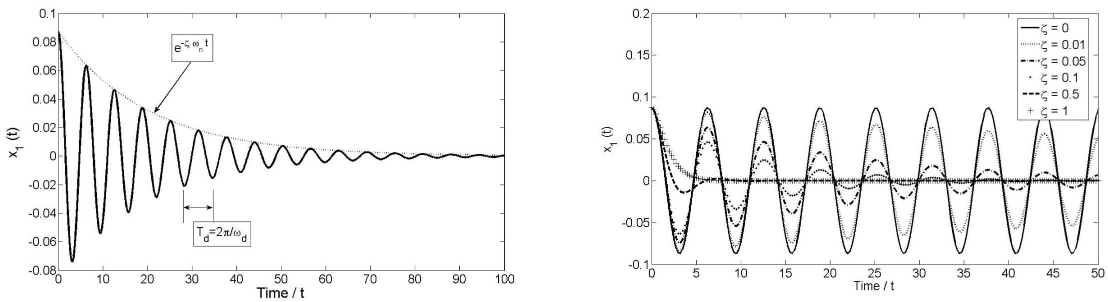
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\phi_{IC} = \arcsin\left(\sqrt{1-\zeta^2}\right)$, and the constants α and r_i are determined by,

$$\alpha = (s+\lambda_i)G(s)U(s)\,\big|_{s=-\lambda_i}\,,$$
$$r_i = (s+\beta_i)G(s)U(s)\,\big|_{s=-\beta_i}\,.$$

4.1.1 Free Vibrations

When the external forcing $f(t)$ is zero, that is $U(s) = 0$, we say that the system is exhibiting *free vibrations* and if not we say that the system is exhibiting a *forced vibration*. In the case of free vibrations we see that $x_f(t) \equiv 0$ and thus the solution corresponds only to the initial condition part $x_{IC}(t)$. Thus in the case of free vibrations the response of the system is

$$x(t) = x_{IC}(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0)}{\omega_d}\cos(\omega_d t - \pi/2) + \frac{x(0)}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi_{IC})\right)1(t), \quad (4.12)$$



(a) For $\zeta = 0.1$ (b) For different values of ζ in the range $0 \leq \zeta \leq 1$

Fig. 4.5 Initial condition response $x_{IC}(t)$ for several different values of ζ in the range $0 \leq \zeta \leq 1$.

when $0 < \zeta < 1$. Note that this is of the form

$$x(t) = x_{IC}(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)1(t), \tag{4.13}$$

where A and ϕ are constants that depend on ζ, ω_n and the initial conditions $x(0), \dot{x}(0)$.

Figure-4.5 shows the plot of $x_{IC}(t)$ for several different values of ζ. The solution exhibits an amplitude decaying oscillatory behavior, of frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ for any initial condition. Thus for $0 < \zeta < 1$ the system said to behave in an *underdamped* manner and ω_d the frequency of oscillation will be referred to as the *damped natural frequency* of the system. Note that the damped free vibration response tends to zero asymptotically as t tends to infinity.

When $\zeta = 0$, that is when there is no damping, the initial condition response is given by

$$x(t) = x_{IC}(t) = \left(\frac{\dot{x}(0)}{\omega_n}\cos(\omega_n t - \pi/2) + x(0)\sin(\omega_n t + \pi/2)\right)1(t), \tag{4.14}$$

which takes the form $x(t) = x_{IC}(t) = A\cos(\omega_n t + \phi)$ and represents an undamped oscillation of frequency ω_n. This motion is referred to as *simple harmonic motion*.

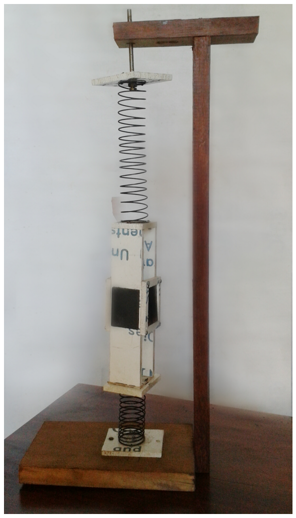
Twice differentiating (4.13) we see that the acceleration of the mass, when $0 < \zeta < 1$, takes the form

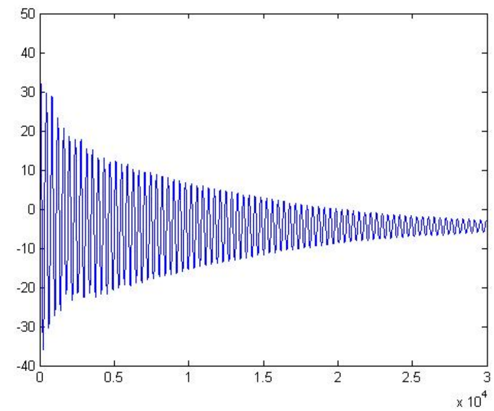
$$\ddot{x}(t) = \ddot{x}_{IC}(t) = A_c e^{-\zeta\omega_n t}\sin(\omega_d t + \phi_c)1(t), \tag{4.15}$$

where A_c and ϕ_c are again constants that depend only on ζ, ω_n and the initial conditions $x(0), \dot{x}(0)$. Comparing this with the experimentally obtained acceleration of the unforced small oscillatory response shown in figure-4.1.1 (b) of the spring mass damper system shown in figure-4.1.1 (a) we see that the physical behaviour of the system matches very well with the theoretically estimated acceleration of the mass that is given by (4.15).

4.1.2 Forced Vibrations and Resonance

In the case of forced vibrations it suffices to look at only the forced response $x_f(t)$ since the initial condition part of the response dies out when damping is positive. We can show that the





(a) Experimental setup of a SMD system (b) Experimentally measured acceleration of the SMD system.

forced response also can be broken up into two parts:

$$x_f(t) = x_{tr}(t) + x_{ss}(t),$$

where

$$x_{tr}(t) = \left(|\alpha|\, e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)\right) 1(t), \tag{4.16}$$

$$x_{ss}(t) = \left(r_1 e^{-\beta_1 t} + \cdots + r_k e^{-\beta_k t}\right)\, 1(t). \tag{4.17}$$

Once again we see that for positive damping the transient part of the response $x_{tr}(t)$ asymptotically tends to zero as t tends to infinity and hence it suffices to only look at the steady state part of the forced response.

Let us consider a very particular type of forcing

$$f(t) = F_0 \cos(\omega t)\, 1(t).$$

Since $u(t) = f(t)/M$ he Laplace transform of $u(t)$ is

$$U(s) = \frac{F_0}{M}\left(\frac{s}{s^2+\omega^2}\right) = \frac{F_0}{M}\left(\frac{s}{(s+i\omega)(s-i\omega)}\right).$$

From (4.17) it can be shown that the steady state response of the system for sinusoidal forcing is

$$x_{ss}(t) = \chi(\omega)\, F_0 \cos(\omega t + \phi(\omega))\, 1(t), \tag{4.18}$$

where

$$\chi(\omega) = \frac{1/M}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega^2\omega_n^2}} \tag{4.19}$$

and

$$\phi(\omega) = \arctan\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right). \tag{4.20}$$

Thus we see that the steady state response to sinusoidal forcing is also sinusoidal with frequency of oscillation equal to that of the forcing frequency ω. However it differs from the forcing in two distinct ways. One is that the amplitude of the steady state solution is $\chi(\omega)$ times the magnitude of the forcing while the other is that the phase of the steady state solution is shifted by $\phi(\omega)$ from the phase of the forcing. It is also important to note that the amplitude magnification and the phase shift depend on the forcing frequency ω. We will see that this observation has crucial implications for vibration analysis and control system design. In summary:

For positively damped ($\zeta > 0$) systems the solution of the system approaches the steady state response. The steady state response to sinusoidal forcing is also sinusoidal with oscillation frequency equal to that of the forcing frequency ω. However the amplitude of the steady state sinusoidal response is $\chi(\omega)$ times the amplitude of the forcing amplitude and the phase of the steady state sinusoidal response is shifted by $\phi(\omega)$ from that of the forcing. The amplification factor $\chi(\omega)$ and the phase shift $\phi(\omega)$ depend on the forcing frequency ω and are given by (4.19) and (4.20) respectively and are plotted in figure-4.6.

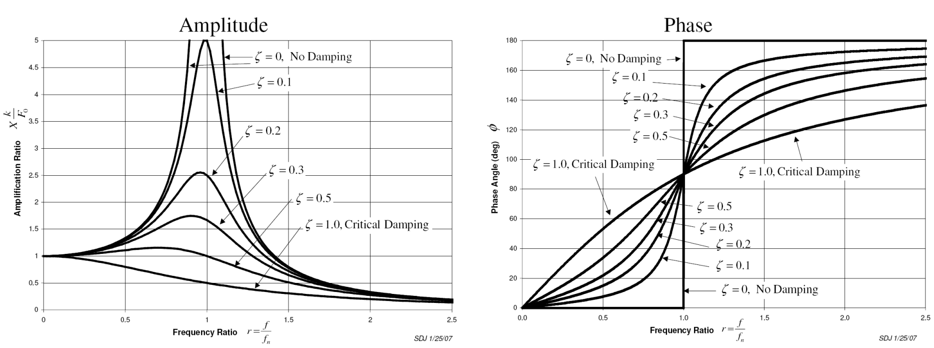


Fig. 4.6 The Frequency response of the Spring Mass Damper system

The important implication of this observation in forced vibration analysis is that the amplitude of the forced response, $\chi(\omega)$, reaches a maximum when the forcing frequency ω is equal to a certain value ω_r. Differentiating (4.19) we find this value to be given by

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}. \tag{4.21}$$

Observe that as the damping becomes negligible the resonance frequency tends to the undamped natural frequency, ω_n. When the forcing frequency reaches the resonance frequency the amplitude of the forced response reaches a maximum value of

$$x_{ssmax} = \frac{F_0/M}{2\zeta\omega_n^2\sqrt{1-\zeta^2}}. \tag{4.22}$$

This phenomena is called *Resonance* and ω_r is called the damped resonance frequency. Observe that when the damping in the system is very small ($\zeta \approx 0$) the amplitude of vibration of the forced system becomes very large and can cause catastrophic damage to the system.

Observe that $\chi(\omega)$ and $\phi(\omega)$ can be written as

$$\chi(\omega) = |G(i\omega)|$$

and

$$\phi(\omega) = \angle G(i\omega),$$

where

$$G(s) = \frac{1/M}{(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

is called the *transfer function* of the spring mass damper system. In this context $G(i\omega)$ is called the frequency response of the system. Its magnitude $\chi(\omega)$ is called the *Magnitude Bode* response and its phase $\phi(\omega)$ is called the *Phase Bode* response of the system. These are plotted in figure-4.6 as a function of the ratio $\omega/\omega_n = f/f_n$.

4.2 A 2-DOF Coupled Vibratory System

In this section we will begin to develop the tools necessary to analyze the vibratory motion of coupled systems such as multi-story buildings, rotary machines mounted on elastic foundations, and multi-rotor shafts. Specifically to keep things simple we will address in detail a system that can be approximated as a 2-DOF coupled spring mass damper system with external forcing.

Consider the coupled 2-DOF spring mass damper system shown in figure-4.7. Choosing the equilibrium positions of the masses as the reference points and applying Newton's laws we obtain

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = f_1(t), \tag{4.23}$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = 0, \tag{4.24}$$

where we have set $f_1(t) = p_0\cos(\omega t)$. We can write (4.23)-(4.24) as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{4.25}$$

Let

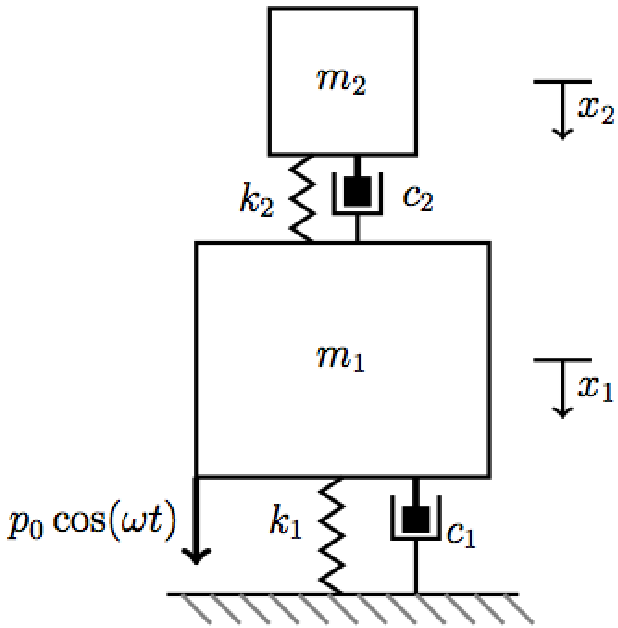


Fig. 4.7 A 2-DOF coupled spring mass damper system

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ f(t) = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix},\ M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\ C = \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\ K = \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}.$$

Then we can write (4.25) as

$$M\ddot{x} + C\dot{x} + Kx = f(t). \tag{4.26}$$

Notice the similarity with the 1-DOF spring mass damper system. We are interested in solving this 2^{nd} order matrix ODE in order to understand the vibratory behavior of the system. We do so in two steps. Recalling the crucial role the undamped natural frequency played in the resonance analysis of the 1-DOF spring mass damper system we first investigate undamped free vibrations of the system and then consider the damped forced vibration behavior of the system.

4.2.1 Frequency Response

Similar to the 1-DOF case we begin by taking a close look at the response of the system by taking the Laplace transform of both sides of (4.26)

$$\mathscr{L}\{M\ddot{x} + C\dot{x} + Kx\} = \mathscr{L}\{f(t)\}. \tag{4.27}$$

Then we have

$$(Ms^2 + Cs + K)X(s) = (Ms + C)x(0) + M\dot{x}(0) + F(s), \tag{4.28}$$

where

$$X(s) = \mathscr{L}\{x(t)\}, \;\; F(s) = \mathscr{L}\{f(t)\}.$$

Let

$$\Delta(s) = (Ms^2 + Cs + K) = \begin{bmatrix} \left(m_1 s^2 + (c_1+c_2)s + (k_1+k_2)\right) & -(c_2 s + k_2) \\ -(c_2 s + k_2) & (m_2 s^2 + c_2 s + k_2) \end{bmatrix}$$

Then from (4.28) we have that

$$X(s) = \Delta(s)^{-1}\left((Ms+C)x(0) + M\dot{x}(0)\right) + \Delta(s)^{-1}F(s),$$

and that

$$x(t) = \mathscr{L}^{-1}\left\{\Delta(s)^{-1}\left((Ms+C)x(0) + M\dot{x}(0)\right)\right\} + \mathscr{L}^{-1}\left\{\Delta(s)^{-1}F(s)\right\}.$$

The first part of the right hand side of the above equation depends only on the initial conditions while the second part depends only on the forcing. Thus similar to the 1-DOF case we can write

$$x(t) = x_{IC}(t) + x_f(t)$$

where

$$x_{IC}(t) = \mathscr{L}^{-1}\left\{\Delta(s)^{-1}\left((Ms+C)x(0) + M\dot{x}(0)\right)\right\}, \tag{4.29}$$

$$x_f(t) = \mathscr{L}^{-1}\left\{\Delta(s)^{-1}F(s)\right\}. \tag{4.30}$$

Similar to the 1-DOF case it can be shown that when all $k_1, k_2, c_1, c_2 > 0$ the initial condition response $x_{IC}(t)$ tends to zero as t tends to infinity. Thus it suffices to only consider the forced response $x_f(t)$. It can be shown that

$$x_f(t) = \mathscr{L}^{-1}\left\{\begin{bmatrix} G_1(s)F_1(s) \\ G_2(s)F_1(s) \end{bmatrix}\right\},$$

where

$$F_1(s) = \mathscr{L}\{f_1(t)\}, \;\; G_1(s) = \frac{m_2 s^2 + c_2 s + k_2}{|\Delta(s)|}, \;\; G_2(s) = \frac{c_2 s + k_2}{|\Delta(s)|},$$

and

$$|\Delta(s)| = \det(\Delta(s)) = \left(m_1 s^2 + (c_1+c_2)s + (k_1+k_2)\right)(m_2 s^2 + c_2 s + k_2) - (c_2 s + k_2)^2. \tag{4.31}$$

It can be shown that when all $k_1, k_2, c_1, c_2 > 0$ the transient part of the forced response asymptotically tends to zero when t tends to infinity and thus the solution tends to the steady state part of the forced response. Hence if $f_1(t) = p_0 \cos(\omega t)$ we have that

$$x_{ss}(t) = \begin{bmatrix} \chi_1(\omega) p_0 \cos(\omega t + \phi_1(\omega)) \\ \chi_2(\omega) p_0 \cos(\omega t + \phi_2(\omega)) \end{bmatrix},$$

where

$$\chi_1(\omega) = |G_1(i\omega)|, \tag{4.32}$$
$$\chi_2(\omega) = |G_2(i\omega)|, \tag{4.33}$$

and

$$\phi_1(\omega) = \angle G_1(i\omega), \tag{4.34}$$
$$\phi_2(\omega) = \angle G_2(i\omega), \tag{4.35}$$

Making the following parameter changes

$$\omega_{n1}^2 = \frac{k_1}{m_1},\ \ \omega_{n2}^2 = \frac{k_2}{m_2},\ \ 2\zeta_1\omega_{n1} = \frac{c_1}{m_1},\ \ 2\zeta_2\omega_{n2} = \frac{c_2}{m_2},\ \ \mu = \frac{m_2}{m_1}$$

we have that the characteristic polynomial (4.31) is

$$|\Delta(s)| = \left(s^4 + 2(\zeta_1\omega_{n1} + \zeta_2\omega_{n2} + \mu\zeta_2\omega_{n2})s^3 + (\omega_{n1}^2 + \omega_{n2}^2 + \mu\omega_{n2}^2 + 4\zeta_1\omega_{n1}\zeta_2\omega_{n2})s^2 + 2(\zeta_2\omega_{n2}\omega_{n1}^2 + \zeta_1\omega_{n1}\omega_{n2}^2)s + \omega_{n1}^2\omega_{n2}^2\right)$$

and hence

$$|\Delta(i\omega)| = \left(\omega^4 - (\omega_{n1}^2 + \omega_{n2}^2 + \mu\omega_{n2}^2 + 4\zeta_1\omega_{n1}\zeta_2\omega_{n2})\omega^2 + \omega_{n1}^2\omega_{n2}^2\right) + 2i\left((\zeta_2\omega_{n2}\omega_{n1}^2 + \zeta_1\omega_{n1}\omega_{n2}^2)\omega - (\zeta_1\omega_{n1} + \zeta_2\omega_{n2} + \mu\zeta_2\omega_{n2})\omega^3\right)$$

$$\chi_1(\omega) = \frac{\frac{1}{m_1}\sqrt{(\omega_{n2}^2 - \omega^2)^2 + 4\zeta_2^2\omega_{n2}^2\omega^2}}{\sqrt{\left(\omega^4 - (\omega_{n1}^2 + \omega_{n2}^2 + \mu\omega_{n2}^2 + 4\zeta_1\omega_{n1}\zeta_2\omega_{n2})\omega^2 + \omega_{n1}^2\omega_{n2}^2\right)^2 + 4\left((\zeta_2\omega_{n2}\omega_{n1}^2 + \zeta_1\omega_{n1}\omega_{n2}^2)\omega - (\zeta_1\omega_{n1} + \zeta_2\omega_{n2} + \mu\zeta_2\omega_{n2})\omega^3\right)^2}},$$

$$\chi_2(\omega) = \frac{\frac{1}{m_1}\sqrt{\omega_{n2}^4 + 4\zeta_2^2\omega_{n2}^2\omega^2}}{\sqrt{\left(\omega^4 - (\omega_{n1}^2 + \omega_{n2}^2 + \mu\omega_{n2}^2 + 4\zeta_1\omega_{n1}\zeta_2\omega_{n2})\omega^2 + \omega_{n1}^2\omega_{n2}^2\right)^2 + 4\left((\zeta_2\omega_{n2}\omega_{n1}^2 + \zeta_1\omega_{n1}\omega_{n2}^2)\omega - (\zeta_1\omega_{n1} + \zeta_2\omega_{n2} + \mu\zeta_2\omega_{n2})\omega^3\right)^2}},$$

Making the further substitution $\omega_{n2} = \alpha\omega_{n1}$ we have

$$\chi_1(\omega) = \frac{\frac{1}{m_1}\sqrt{(\alpha^2\omega_{n1}^2 - \omega^2)^2 + 4\zeta_2^2\alpha^2\omega_{n1}^2}}{\sqrt{\left(\omega^4 - (1 + \alpha^2 + \mu\alpha^2 + 4\alpha\zeta_1\zeta_2)\omega_{n1}^2\omega^2 + \alpha^2\omega_{n1}^4\right)^2 + 4\left((\zeta_2\alpha + \zeta_1\alpha^2)\omega_{n1}^3\omega - (\zeta_1 + \zeta_2\alpha + \mu\zeta_2\alpha)\omega_{n1}\omega^3\right)^2}},$$

$$\chi_2(\omega) = \frac{\frac{1}{m_1}\alpha\omega_{n1}\sqrt{\alpha^2\omega_{n1}^2 + 4\zeta_2^2\omega^2}}{\sqrt{\left(\omega^4 - (1 + \alpha^2 + \mu\alpha^2 + 4\alpha\zeta_1\zeta_2)\omega_{n1}^2\omega^2 + \alpha^2\omega_{n1}^4\right)^2 + 4\left((\zeta_2\alpha + \zeta_1\alpha^2)\omega_{n1}^3\omega - (\zeta_1 + \zeta_2\alpha + \mu\zeta_2\alpha)\omega_{n1}\omega^3\right)^2}},$$

The frequency response $X_1(\omega)$ of the coupled spring mass damper system when $\alpha = 2$ and $\mu = 0.5$ is plotted in figure-4.8 for several low values of the damping ratios.

If the damping is negligible we have $\zeta_1, \zeta_2 \approx 0$ and thus

$$\chi_1(\omega) = \frac{(\omega_{n2}^2 - \omega^2)/m_1}{\left(\omega^4 - (1 + \alpha^2 + \mu\alpha^2)\omega_{n1}^2\omega^2 + \alpha^2\omega_{n1}^4\right)}, \tag{4.36}$$

$$\chi_2(\omega) = \frac{\omega_{n2}^2/m_1}{\left(\omega^4 - (1 + \alpha^2 + \mu\alpha^2)\omega_{n1}^2\omega^2 + \alpha^2\omega_{n1}^4\right)}. \tag{4.37}$$

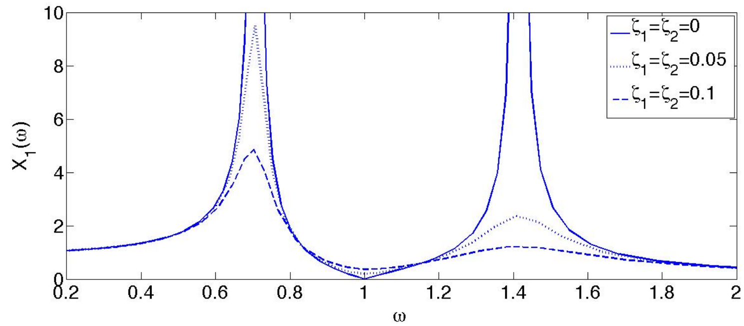


Fig. 4.8 The frequency response of the coupled spring mass damper system when $\omega_{n1} = 1$, $\alpha = 2$ and $\mu = 0.5$.

The denominator of $\chi_1(\omega)$ and $\chi_2(\omega)$ is the characteristic polynomial of the system with s replaced by $i\omega$

$$|\Delta(\omega)| = \left(\omega^4 - (1+\alpha^2+\mu\alpha^2)\omega_{n1}^2\,\omega^2 + \alpha^2\omega_{n1}^4\right). \tag{4.38}$$

The roots of this polynomial are given by

$$\omega_1 = \omega_{n1}\left(\frac{(1+\alpha^2+\mu\alpha^2) - \sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}}{2}\right)^{\frac{1}{2}}, \tag{4.39}$$

$$\omega_2 = \omega_{n1}\left(\frac{(1+\alpha^2+\mu\alpha^2) + \sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}}{2}\right)^{\frac{1}{2}}. \tag{4.40}$$

Observe that $\omega_1 < \omega_2$. Thus we see that the denominators of $\chi_1(\omega)$ and $\chi_2(\omega)$ become infinite when the driving frequency is equal to $\omega = \omega_1$ and $\omega = \omega_2$. Thus ω_1 and ω_2 are called the undamped resonance frequencies of the system.

From (4.36) we notice that the steady state amplitude reaches zero when the forcing frequency satisfies $\omega = \omega_{n2}$. This observation will play a crucial role in vibration mitigation in systems to be investigated in the next section.

4.2.1.1 Tuned Mass Damper Vibration Absorber Design

Let us consider the problem of resonance mitigation in a machine with an unbalanced rotary component. The unbalance will give rise to periodic forcing on the mountings of the machine with frequency equal to the rotational frequency of the machine. The operating conditions of the machine may require that the frequency take values in a certain finite range of frequencies. If the natural frequency of the system comprising the machine plus mounts falls in this range, the machine will exhibit large amplitude vibrations when the machine operating frequency is close to this natural frequncy due to resonance. How does one devise a method to reduce this large amplitude of vibration for the entire range of operating frequencies of the machine?

On a first approximation, the machine acted on by this forcing can be modeled as a simple 1-DOF spring mass system on which a force $f(t)$ is acting. We neglect damping for the

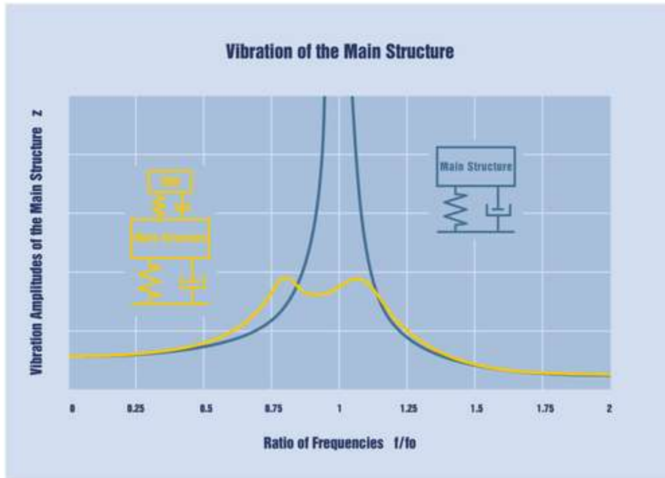


Fig. 4.9 The concept of a vibration absorber.

moment. Recall the conclusion made at the end of the previous section. There we saw that if we connect another spring mass system with natural frequency close to that of the machine then the amplitude of vibration of the machine becomes zero. However the addition of this second system makes the composite system a 2-DOF system. Figure-4.9 shows the frequency response of such a setup. The composite 2-DOF has two modes of vibration and hence two resonance frequencies. If these two resonance frequencies fall inside the operating frequencies of the machine resonance will occur now at two operating conditions. Thus the challenge is to design this added system so that the natural frequencies of the composite system lie well outside the operating region of the machine. We analyze this scenario a bit further using (4.36) and (4.37). Let ω_{n1} be the natural frequency of the machine and let ω_{n2} be the natural frequency of the added system.

Let $\gamma = \omega_{n1}/\omega$. Then (4.36) and (4.37) can be written as

$$\chi_1(\gamma) = \frac{(\alpha^2\gamma^2 - 1)/m_1}{(1-(1+\alpha^2+\mu\alpha^2)\gamma^2+\alpha^2\gamma^4)}, \tag{4.41}$$

$$\chi_2(\gamma) = \frac{\alpha^2\gamma^2/m_1}{(1-(1+\alpha^2+\mu\alpha^2)\gamma^2+\alpha^2\gamma^4)}, \tag{4.42}$$

Furtermore from (4.123) and (4.123) we have

$$\omega_2^2 - \omega_1^2 = \omega_{n1}^2\sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}. \tag{4.43}$$

From (4.41) we find that the steady state amplitude reaches zero when $\alpha^2\gamma^2 = 1$. That is when $\omega = \omega_{n2} = \alpha\omega_{n1}$. Moreover the plot of (4.41) shows that the amplitude of vibration of the main mass m_1 remains small within a certain frequency range that falls between the interval $\omega_1 < \omega < \omega_2$. Thus tuned mass damper design objective reduces to that of finding α such that the range of operational frequencies of the machine fall well within the interval $\omega_1 < \omega < \omega_2$. One way of doing this is by trying to maximize $\omega_2 - \omega_1$ while ensuring that

$\omega_{n2} = \alpha\omega_{n1}$ falls in the middle of the interval between ω_2 and ω_1. That is we try to find α such that $\omega_{n2} = \alpha\omega_{n1} = (\omega_2 + \omega_1)/2$ and $\omega_2 - \omega_1$ is maximized.

From (4.43) we see that

$$(\omega_2 - \omega_1)(\omega_2 + \omega_1) = (\omega_2 - \omega_1)2\alpha\omega_{n1} = \omega_{n1}^2\sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}\,.$$

and hence

$$(\omega_2 - \omega_1) = \frac{\omega_{n1}\sqrt{\left(\frac{1}{\alpha} + (1+\mu)\alpha\right)^2 - 4}}{2}\,.$$

For large α

$$(\omega_2 - \omega_1) \approx \frac{\omega_{n1}(1+\mu)\alpha}{2}\,.$$

Thus we see that $(\omega_2 - \omega_1)$ can be maximized by maximizing α and μ. However from (4.42) we see that increasing α increases the amplitude of vibration of the added mass m_2 and thus requiring a compromise as typical in any engineering design problem. For practical purposes we also desire to choose μ as small as possible.

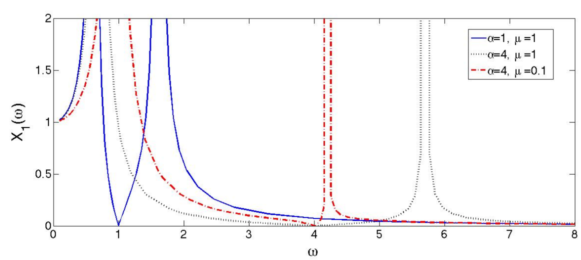


Fig. 4.10 The amplitude magnification $X_1(\omega)$ for several combinations of α and μ.

MATLAB Simulations

You may use the following M-files to generate the frequency response and the time response of a coupled 2-DOF spring mass damper system.

Frequency Response

```
function [Mag,Ph,W]=FrequancyResponseCoupledSMD
z1=0; %zeta_1
z2=0; %zeta_2
w1=1; %omega_n1
w2=4; %omega_n2
m1=1; %m_1
mu=.1; %mu
```

```
al=w2/w1; %alpha

num=[1 2*z2*w2 w2^2];
den=[1 2*(z1*w1+z2*w2+mu*z2*w2) (w1^2+w2^2+mu*w2^2+4*z1*z2*w1*w2) 2*(z2*
[Mag,Ph,W]=bode(num,den);
plot(W,Mag)
end
```

Time Response

```
function [T,Y]=CoupledSpringMassDamperSystem(Tmax)
z1=0.1; % zeta_1
z2=0.1; % zeta_2
w1=1;   % omega_n1
w2=2;   %omega_n2
m1=1;   %m_1
mu=.5;  %mu
al=w2/w1; % alpha
x1bar=(-1+al^2-mu*al^2+sqrt((1+al^2+mu*al^2)^2-4*al^2))/(2*al^2);

x0(1:2)=[x1bar;1];
x0(3:4)=[0;0];

[T,Y]=ode45(@CoupledSMDEqns,[0 Tmax],x0);
plot(T,Y(:,1:2))

function Xdot=CoupledSMDEqns(t,x)
    X=x(1:2);
    V=x(3:4);

    f1=0;
    C=[(2*z1*w1+mu*2*z2*w2) -mu*2*z2*w2;-2*z2*w2 2*z2*w2];
    K=[(w1^2+mu*w2^2) -mu*w2^2;-w2^2 w2^2];

    xdot=V;
    vdot=-C*V-K*X+[f1/m1;0];

    Xdot=[xdot;vdot];
end
end
```

4.2.2 Modal Analysis

In vibration analysis we are typically interested in near resonant conditions. That is, in cases where the system is very lightly damped and the forcing frequency is close to the resonance frequencies of the system. Since the behavior of the system depends smoothly on the damping ratios we see that the real behavior of the system will thus be described in a qualitative and approximately quantitative sense quite well by analyzing the case where the damping is zero. Thus in this section we will assume that the damping is negligible and hence that the system equations (4.26) are approximately given by

$$M\ddot{x}+Kx=f(t), \tag{4.44}$$

where $f(t)=[f_1(t)\ \ 0]^T$. Multiplying the above equation by M^{-1} we have

$$\ddot{x}+\Omega x=M^{-1}f(t), \tag{4.45}$$

where $\Omega=M^{-1}K$. Recall that, for notational convenience, in the previous section we had set

$$\omega_{n1}^2=\frac{k_1}{m_1},\ \ \omega_{n2}^2=\frac{k_2}{m_2},\ \ \mu=\frac{m_2}{m_1},\ \ \alpha=\frac{\omega_{n2}}{\omega_{n1}}.$$

Then we have

$$\Omega=\begin{bmatrix}1/m_1 & 0\\ 0 & 1/m_2\end{bmatrix}\begin{bmatrix}(k_1+k_2) & -k_2\\ -k_2 & k_2\end{bmatrix}=\begin{bmatrix}\frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1}\\ -\frac{k_2}{m_2} & \frac{k_2}{m_2}\end{bmatrix}=\begin{bmatrix}(1+\alpha^2\mu)\omega_{n1}^2 & -\mu\alpha^2\omega_{n1}^2\\ -\alpha^2\omega_{n1}^2 & \alpha^2\omega_{n1}^2\end{bmatrix}.$$

Also recall from the previous section that the resonance frequencies correspond to the roots of the characteristic polynomial of the system (4.38)

$$|\Delta(\omega)|=\left(\omega^4-(1+\alpha^2+\mu\alpha^2)\omega_{n1}^2\,\omega^2+\alpha^2\omega_{n1}^4\right),$$

and are given by (4.123) and (4.123)

$$\omega_1=\omega_{n1}\left(\frac{(1+\alpha^2+\mu\alpha^2)-\sqrt{(1+\alpha^2+\mu\alpha^2)^2-4\alpha^2}}{2}\right)^{\frac{1}{2}},$$

$$\omega_2=\omega_{n1}\left(\frac{(1+\alpha^2+\mu\alpha^2)+\sqrt{(1+\alpha^2+\mu\alpha^2)^2-4\alpha^2}}{2}\right)^{\frac{1}{2}}.$$

Notice that $|\Delta(\omega)|=\det(-\omega^2I+\Omega)$ and hence that the resonance frequencies ω_1 and ω_2 are the square roots of the eigenvalues of the matrix Ω.

Let us investigate what the corresponding eigenvectors $\bar{x}$ tell us. The eigenvectors are given by the nontrivial solutions of

$$(-\omega_i^2 I + \Omega)\bar{x}_i = 0. \tag{4.46}$$

Recall that the eigenvectors are only unique up to a scalar multiplication. It is thus customary to consider the normalized version in the analysis to avoid ambiguity. Since $\bar{x}_1, \bar{x}_2$ are the eigenvectors of Ω we see that by definition $\Omega\bar{x}_1 = \omega_1^2\bar{x}_1$ and $\Omega\bar{x}_2 = \omega_2^2\bar{x}_2$. and hence if $T = [\bar{x}_1 \quad \bar{x}_2]$ then $\Omega T = TD$ where $D = \text{diag}\{\omega_1^2, \omega_2^2\}$ is a diagonal matrix explicitly given by

$$D = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = T^{-1}\Omega T.$$

Consider the coordinate transformation $x = Tz$ where $z = [z_1 \quad z_2]^T$. Substituting in (4.45) we obtain

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T^{-1} \begin{bmatrix} \frac{1}{m_1} f_1(t) \\ 0 \end{bmatrix},$$

and hence that in these new coordinates, z, the coupled pair of ODEs given by (4.45) reduce to the uncoupled pair of equations

$$\ddot{z}_1 + \omega_1^2 z_1 = \gamma_1 f_1(t), \tag{4.47}$$

$$\ddot{z}_2 + \omega_2^2 z_2 = \gamma_2 f_1(t), \tag{4.48}$$

where we have denoted $\gamma_1 = \frac{T^{11}}{m_1}$ and $\gamma_2 = \frac{T^{21}}{m_1}$ with

$$T^{-1} = \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix}.$$

These equations are referred to as the normalized equations of the system (4.45). Since $x = Tz$ notice that

$$x(t) = \bar{x}_1 z_1(t) + \bar{x}_2 z_2(t).$$

and hence the response of the system $x(t)$ is in fact a linear combination of the responses of the two uncoupled oscillators $z_1(t)$ and $z_2(t)$. Thus the first equation corresponds to the first mode, (4.123), while the second equation corresponds to the second mode, (4.123).

In summary this shows that the coordinate transformation $x = Tz$ where the columns of T are the eigenvectors of Ω has transformed the coupled system (4.45) to the pair of uncoupled two 1-DOF spring mass systems (4.47) and (4.48). The response of the system is a linear combination of the response of these two uncoupled systems that we refer to as modes. Furthermore notice that even though the external force was only acting on the first coupled oscillator in the physical system (i.e. mass m_1), it is felt by both uncoupled oscillators in these new coordinates.

Let $\bar{x}_1 = [\bar{x}_{11} \quad \bar{x}_{21}]^T$ and $\bar{x}_2 = [\bar{x}_{12} \quad \bar{x}_{22}]^T$ denote the normalized eigenvectors of Ω that correspond to the two eigenvalues ω_1^2 and ω_2^2 respectively. Then from (4.46) we find their components to explicitly satisfy

$$r_1 \triangleq \frac{\bar{x}_{11}}{\bar{x}_{21}} = \frac{\alpha^2 - 1 - \mu\alpha^2 + \sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}}{2\alpha^2} > 0, \tag{4.49}$$

$$r_2 \triangleq \frac{\bar{x}_{12}}{\bar{x}_{22}} = \frac{\alpha^2 - 1 - \mu\alpha^2 - \sqrt{(1+\alpha^2+\mu\alpha^2)^2 - 4\alpha^2}}{2\alpha^2} < 0. \tag{4.50}$$

Then from (4.51) we see

$$x(t) = \bar{x}_1 z_1(t) + \bar{x}_2 z_2(t) = \bar{x}_{21} \begin{bmatrix} r_1 \\ 1 \end{bmatrix} z_1(t) + \bar{x}_{22} \begin{bmatrix} r_2 \\ 1 \end{bmatrix} z_2(t). \tag{4.51}$$

In the following we will consider two cases the unforced case: free vibrations, and the forced case: forced vibrations.

4.2.2.1 Free Vibrations

In this section we will consider the case where the forcing is zero. That is $f_1(t) \equiv 0$. Then (4.47) and (4.48)

$$\ddot{z}_1 + \omega_1^2 z_1 = 0$$
$$\ddot{z}_2 + \omega_2^2 z_2 = 0.$$

The initial condition solution of this unforced system takes the form

$$z_1(t) = A_1 \cos(\omega_1 t + \phi_1),$$
$$z_2(t) = A_2 \cos(\omega_2 t + \phi_2),$$

where A_1, A_2, ϕ_1 and ϕ_2 are constants that are uniquely determined by the initial conditions $z_1(0), z_2(0)$ and $\dot{z}_1(0), \dot{z}_2(0)$. Then from (4.51) we see that the general solution takes the form

$$x(t) = A_1 \bar{x}_1 \cos(\omega_1 t + \phi_1) + A_2 \bar{x}_2 \cos(\omega_2 t + \phi_2)$$

$$= A_1 \begin{bmatrix} \bar{x}_{11} \\ \\ \bar{x}_{21} \end{bmatrix} \cos\left(\omega_1 t + \phi_1\right) + A_2 \begin{bmatrix} \bar{x}_{11} \\ \\ \bar{x}_{21} \end{bmatrix} \cos\left(\omega_2 t + \phi_2\right).$$

This shows us that when $x(0) \approx \bar{x}_1, \dot{x}(0) = 0$ then

$$x(t) \approx \bar{x}_1 \cos\left(\omega_1 t\right)$$

while when $x(0) \approx \bar{x}_2, \dot{x}(0) = 0$ then

$$x(t) \approx \bar{x}_2 \cos\left(\omega_2 t\right).$$

In the firs case both masses are oscillating in synchrony at a frequency ω_1 while in the second case both masses are oscillating in synchrony at a frequency ω_2. Furthermore in the first case if the second mass moves one unit then the first mass moves r_1 units in the same direction, where $r_1 > 0$ is given by (4.123), while in the second case if the second mass moves one unit then the first mass moves $|r_2|$ units in the opposite direction, where $r_2 < 0$ is given by (4.123). The response $\bar{x}_1 \cos\left(\omega_1 t\right)$ is called the first mode of vibration while $\bar{x}_2 \cos\left(\omega_2 t\right)$ is called the second mode. These two responses are plotted in figure-4.11 for the case of $\mu = 0.5$ and $\alpha = 2$. Relating these to the frequency response of the system shown in figure-4.8 we highlight that the first mode corresponds to the first resonance condition while the second mode corresponds to the second resonance condition.

4.2.2.2 Forced Vibrations and Resonance

In this section we consider the case where a sinusoidal forcing of $f_1(t) = f_0 \cos \omega t$ is applied at $t = 0$ when the system is in rest and look at the steady state part of the response. From (4.47), (4.48) we see that each of the two modes are excited by $f_0 \cos \omega t$. Thus we see from (4.18) that the steady state solutions take the form

$$\begin{aligned} z_1(t) &= f_0 \chi_1(\omega) \cos\left(\omega t + \phi_1(\omega)\right), \\ z_2(t) &= f_0 \chi_2(\omega) \cos\left(\omega t + \phi_2(\omega)\right), \end{aligned}$$

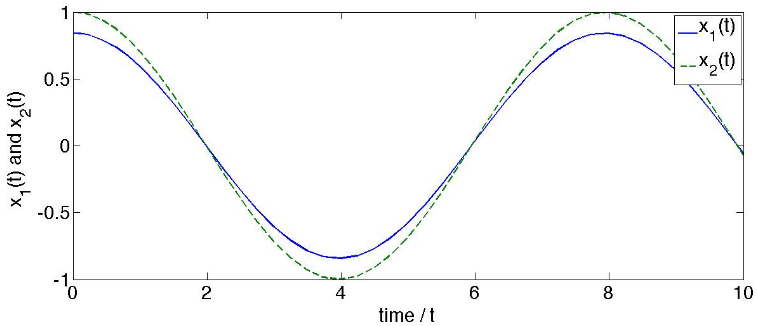
where for $j = 1,2$ the frequency responses are of the two uncoupled oscillators are given by $\chi_j(\omega) = |G_j(\omega)|$ and $\phi_i(\omega) = \angle G_j(\omega)$ where

$$G_j(\omega) = \frac{\gamma_j}{-\omega^2 + \omega_j^2}.$$

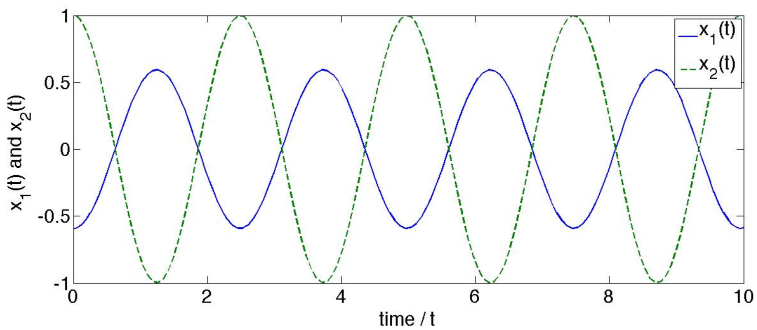
This shows us that if the forcing frequency is close to ω_j, the frequency corresponding to the j^{th} mode, then only $z_j(t)$ will have a very large amplitude compared to the other mode $z_k(t)$. That is since $x(t) = \bar{x}_1 z_1(t) + \bar{x}_2 z_2(t)$ we see that when $\omega \approx \omega_1$

$$x(t) \approx \bar{x}_1 z_1(t) = f_0 \chi_1(\omega) \cos\left(\omega t + \phi_1(\omega)\right) \bar{x}_1$$

while when $\omega \approx \omega_2$ then



(a) The first mode of vibration with vibrational frequency ω_1.



(b) The second mode of vibration with vibrational frequency ω_2.

Fig. 4.11 The two modes of free vibration of the coupled spring mass system for $\alpha = 2$ and $\mu = 0.5$.

$$x(t) \approx \bar{x}_2 z_2(t) = f_0 \chi_2(\omega) \cos(\omega t + \phi_2(\omega)) \bar{x}_2.$$

We invite the reader to compare these results with the frequency response of the system shown in figure-4.8.

In summary we can conclude that when the forcing frequency is close to any of the frequencies corresponding to a mode we will observe a large amplitude motion of that mode. Thats is when the forcing frequency ω is close to ω_j the steady state response of the system $x(t)$ takes the shape of the j^{th} mode $\bar{x}_j$ and will be oscillating in synchrony at a frequency close to $\omega \approx \omega_j$.

In vibration problems one is mainly interested in conditions near resonance and hence in the case where damping is negligibly small. Since the behavior of the system depends smoothly on the damping ratios we see that the real behavior of the system is describe in a qualitative and approximately quantitative sense quite well by the above analysis.

We conclude by noting that in general the steady state response solution takes the form

$$x(t) = f_0 \chi_1(\omega) \cos(\omega t + \phi_1(\omega)) \bar{x}_1 + f_0 \chi_2(\omega) \cos(\omega t + \phi_2(\omega)) \bar{x}_2.$$

4.2.3 Beating

In this section we will investigate closely a phenomena called beating. To do so we consider the Wilberforce oscillatory system in the applied mechanics lab. A schematic of the setup is shown in Figure: 4.12. The spring uncoils as it stretches and coils as it compresses. This gives rise to a coupling of the lateral motion to the torsional motion of the mass. It can be shown that, if the damping in the system is negligible and the displacements are small then the motion of the system is sufficiently accurately described by the solutions of the coupled set of equations given by

$$M\ddot{x}+k_x x+\varepsilon\theta=0,$$
$$I\ddot{\theta}+k_\theta\theta+\varepsilon x=0,$$

where ε is the coupling parameter and is assumed to be very small (ie. $\varepsilon << 1$). Assume that $k_x/M \approx k_\theta/I$.

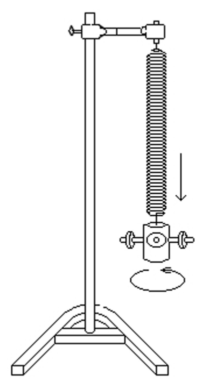


Fig. 4.12 Wilberforce Vibratory System

Writing down the above coupled equations in matrix form we have

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}+\begin{bmatrix}\omega_x^2 & \varepsilon_M\\ \varepsilon_I & \omega_\theta^2\end{bmatrix}\begin{bmatrix}x\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

where $\omega_x^2=\frac{k}{M}$, $\omega_\theta^2=\frac{k}{I}$, $\varepsilon_M=\frac{\varepsilon}{M}$ and $\varepsilon_I=\frac{\varepsilon}{I}$.

Resonance frequencies of the system are given by the square root of the eigenvalues of

$$\Omega=\begin{bmatrix}\omega_n^2 & \varepsilon_M\\ \varepsilon_I & \omega_n^2\end{bmatrix}.$$

That is by the solutions of the characteristic equation

$$\det\left(\begin{bmatrix}(\omega_x^2-\omega^2) & \varepsilon_M\\ \varepsilon_I & (\omega_\theta^2-\omega^2)\end{bmatrix}\right)=\omega^4-(\omega_x^2+\omega_\theta^2)\omega^2+(\omega_x^2\omega_\theta^2-\varepsilon_I\varepsilon_M)=0.$$

When $k/M \approx k/I$, as we observe for the device in the lab, we have $\omega_x^2 \approx \omega_\theta^2 = \omega_n^2$ and hence

$$\omega^4-2\omega_n^2\omega^2+(\omega_n^4-\varepsilon_n^2)=(\omega^2-\omega_n^2)^2-\varepsilon_n^2=0,$$

where $\varepsilon_n^2=\varepsilon_I\varepsilon_M$. The ω that satisfy this expression are explicitly given by

$$\omega_1=\sqrt{\omega_n^2-\varepsilon_n},\quad \omega_2=\sqrt{\omega_n^2+\varepsilon_n},$$

These are the two resonant frequencies of the system. Let $\bar{Y}_1=[\bar{x}_1\ \bar{\theta}_1]^T$ and $\bar{Y}_1=[\bar{x}_1\ \bar{\theta}_1]^T$ be the two corresponding eigenvectors that satisfy

$$\begin{bmatrix}(\omega_x^2-\omega_j^2) & \varepsilon_M\\ \varepsilon_I & (\omega_\theta^2-\omega_j^2)\end{bmatrix}\begin{bmatrix}\bar{x}_j\\ \bar{\theta}_j\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

As in section 4.2.2 we note that by definition $\Omega\bar{Y}_1=\omega_1^2\bar{Y}_1$ and $\Omega\bar{Y}_2=\omega_2^2\bar{Y}_2$. Thus if $T=[\bar{Y}_1\ \bar{Y}_2]$ then $\Omega T=TD$ where

$$D=\begin{bmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{bmatrix}.$$

Hence if we use the new variable $Y=TZ$, where $Z=[Z_1,\ Z_2]^T$ then the equation $\ddot{Y}+\Omega Y=0$ becomes $T\ddot{Z}+\Omega TZ=0$ and hence $T\ddot{Z}+TDZ=0$ and finally by pre-multiplying by T^{-1} we have $\ddot{Z}+DZ=0$. This gives the two uncoupled oscillators

$$\begin{aligned}\ddot{Z}_1+\omega_1^2Z_1&=0,\\ \ddot{Z}_2+\omega_2^2Z_2&=0.\end{aligned}$$

The solution to these equations take the form $Z_1(t)=\alpha_1\cos(\omega_1t+\phi_1)$ and $Z_2(t)=\alpha_2\cos(\omega_2t+\phi_2)$ where $\alpha_1,\phi_1,\alpha_2,\phi_2$ depend on the initial conditions.

From $Y=TZ$ we thus see that any solution of the system takes the form

$$Y(t)=\alpha_1\bar{Y}_1\cos(\omega_1t+\phi_1)+\alpha_2\bar{Y}_2\cos(\omega_2t+\phi_2)$$

where the four unknowns $\alpha_1,\phi_1,\alpha_2,\phi_2$ are uniquely determined by the initial conditions of the system $x(0),\theta(0),\dot{x}(0),\dot{\theta}(0)$. Given these four initial conditions the 4 arbitrary constant can be uniquely determined.

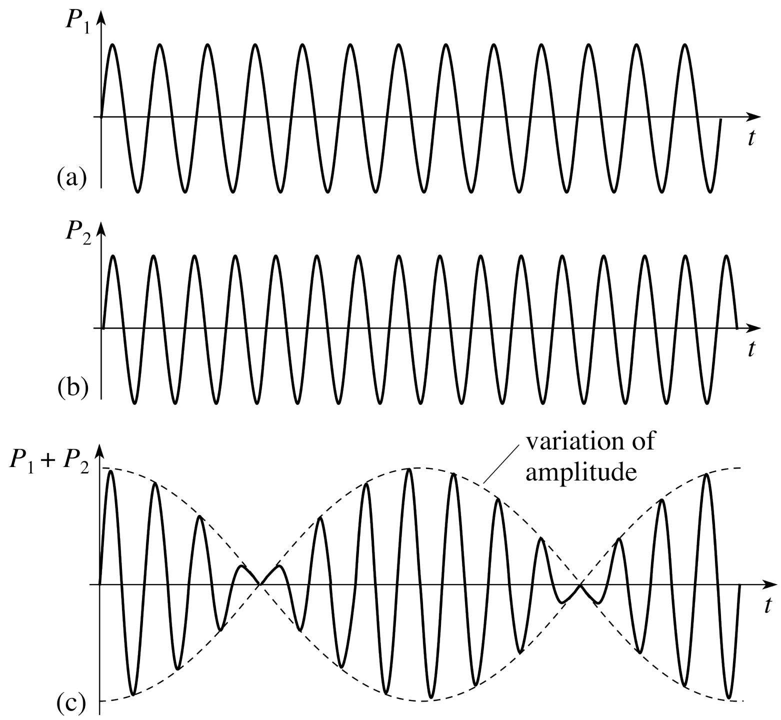


Fig. 4.13 The form of the solution $Y(t)$.

For small coupling the binomial expansion gives $\omega_1 \approx (\omega_n - \frac{\varepsilon_n}{2\omega_n})$ and $\omega_2 \approx (\omega_n + \frac{\varepsilon_n}{2\omega_n})$. Let $\omega_B = \frac{\varepsilon_n}{2\omega_n}$. Then

$$
\begin{aligned}
Y(t) &\approx A_1\bar{Y}_1 \cos(\omega_n t - \omega_B t + \phi_1) + A_2\bar{Y}_2 \cos(\omega_n t + \omega_B t + \phi_2) \\
&= A_1\bar{Y}_1 \left(\cos(\omega_n t)\cos(\omega_B t - \phi_1) + \sin(\omega_n t)\sin(\omega_B t - \phi_1)\right), \\
&\quad + A_2\bar{Y}_2 \left(\cos(\omega_n t)\cos(\omega_B t + \phi_2) - \sin(\omega_n t)\sin(\omega_B t + \phi_2)\right) \\
&= \left(A_1\bar{Y}_1 \cos(\omega_B t - \phi_1) + A_2\bar{Y}_2 \cos(\omega_B t + \phi_2)\right)\cos(\omega_n t) \\
&\quad + \left(A_1\bar{Y}_1 \sin(\omega_B t - \phi_1) - A_2\bar{Y}_2 \sin(\omega_B t + \phi_2)\right)\sin(\omega_n t) \\
&\sim A\cos(\omega_B t + \phi)\cos(\omega_n t + \psi)
\end{aligned}
$$

Figure-4.13 shows the response of $Y(t) = A\cos(\omega_B t + \phi)\cos(\omega_n t + \psi)$. This phenomena is referred to as *beating*.

4.3 n-DOF Undamped Vibration Analysis

In this section we generalize the modal analysis procedure that was used in section -4.2.2 for the 2-DOF case.

The linearized model of a n-DOF undamped vibrational system can be written down as

$$M\ddot{x}+Kx=f(t), \tag{4.52}$$

where M and K are $n\times n$ positive definite matrices and x is a $n\times 1$ column matrix. Typically M is a positive definite symmetric matrix and hence is invertible. Multiplying through by M^{-1} we have

$$\ddot{x}+\Omega x=M^{-1}f(t), \tag{4.53}$$

where $\Omega=M^{-1}K$. Typically Ω is a tri-diagonal positive definite or positive semidefinite matrix. Since Ω is tri-diagonal (4.53) represents a set of n number of coupled second order differential equations.

4.3.1 Free Vibrations

Let us first consider the unforced case. That is the case where $f(t)\equiv 0$.

$$\ddot{x}+\Omega x=0, \tag{4.54}$$

The unforced system (4.54) will have a synchronized solution of the form $x_m(t)=\bar{x}_m\cos(\omega t+\phi)$ if and only if it satisfies the linear second order equation (4.54). Substituting $x_m(t)=\bar{x}_m\cos(\omega t+\phi)$ in (4.54) we have

$$(-\omega^2 I+\Omega)\bar{x}_m\cos(\omega t+\phi)=0.$$

Since $\cos(\omega t+\phi)$ is not identically zero this equation is satisfied if an only if there exists $\bar{x}_m\neq 0$ such that

$$(-\omega^2 I+\Omega)\bar{x}_m=0.$$

There exists $\bar{x}_m\neq 0$ such that the above equation is satisfied only when the matrix $(-\omega^2 I+\Omega)$ is singular. That is only when $\det(-\omega^2 I+\Omega)=0$. This is true when ω^2 equals the eigenvalues of Ω. The solutions $\bar{x}$ are the corresponding eigenvectors of Ω. For typical vibrational problems the eigenvalues of Ω are real, non-negative and distinct. Let these n eigenvalues be $0\leq\omega_1^2<\omega_2^2<\cdots<\omega_n^2$. Since eigenvalues are distinct and real the corresponding eigenvectors are real and form a linearly independent spanning set. Denote these corresponding normalized eigenvectors by $\bar{x}_{m1},\bar{x}_{m2},\cdots,\bar{x}_{mn}$. Thus we have shown that

$$x_{mk}(t)=\bar{x}_{mk}\cos(\omega_k t+\phi_k)$$

is a solution of (4.53) with $f(t)\equiv 0$ for all $i=1,2,\cdots,n$. Such solutions are called the *modes of vibration* of the system. Since the system (4.54) is linear we see that any linear combination of the above modes is also a solution. The question we need to answer is if any general solution of (4.54) can be expressed as a linear combination of the modes. That is if any given solution can be uniquely expressed in the form

$$x(t) = \sum_{k=1}^{n} \alpha_k \bar{x}_{mk} \cos(\omega_k t + \phi_k) \tag{4.55}$$

for some set of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\phi_1, \phi_2, \cdots, \phi_n$. A specific solution of (4.54) is uniquely determined by the complete specification of the $2n$ number of initial conditions $x(0)$ and $\dot{x}(0)$. Thus let us see if we can uniquely determine the $2n$ number of arbitrary constants α_k and ϕ_k for $i = 1, 2, \cdots, n$ in terms of $x(0)$ and $\dot{x}(0)$. Differentiating the above expression we see that

$$\dot{x}(t) = \sum_{k=1}^{n} -\omega_k \alpha_k \bar{x}_{mk} \sin(\omega_k t + \phi_k).$$

Thus we have that

$$x(0) = \sum_{k=1}^{n} \alpha_k \bar{x}_{mk} \cos(\phi_k),$$

$$\dot{x}(0) = \sum_{k=1}^{n} -\omega_k \alpha_k \bar{x}_{mk} \sin(\phi_k).$$

For simplicity let us write these down as

$$x(0) = \sum_{k=1}^{n} \beta_k \bar{x}_{mk}, \tag{4.56}$$

$$\dot{x}(0) = \sum_{k=1}^{n} \gamma_k \bar{x}_{mk}, \tag{4.57}$$

where $\beta_k = \alpha_k \cos(\phi_k)$ and $\gamma_k = -\omega_k \alpha_k \sin(\phi_k)$.

To find these arbitrary constants we observe that the $\{\bar{x}_{m1}, \cdots, \bar{x}_{mn}\}$ are linearly independent to each other. Thus we see that the matrix $T = [\bar{x}_{m1} \quad \cdots \quad \bar{x}_{mn}]$ is invertible. Notice that then (4.56) and (4.57) can then be written as $x(0) = T\beta$ and $\dot{x}(0) = T\gamma$, where $\beta = [\beta_1 \quad \cdots \quad \beta_n]^T$ and $\gamma = [\gamma_1 \quad \cdots \quad \gamma_n]^T$. Thus we see that the $2n$ number of unknown constants β_k and γ_k are uniquely determined by the expressions $\beta = T^{-1}x(0)$ and $\gamma = T^{-1}\dot{x}(0)$ that depend on the initial conditions. This shows that any solution of the unforced system can be uniquely expressed as a linear combination of the modes as in (4.80).

Denote by $\mathscr{X}$ the space of all solutions of the linear ODE (4.54). From linearity it follows that if $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of the system, that is if $x_1(t), x_2(t) \in \mathscr{X}$, then $\alpha x_1(t) + \beta x_2(t) \in \mathscr{X}$ for all $\alpha, \beta \in \mathbb{R}$. That is any linear combination of $x_1(t)$ and $x_2(t)$ is also a solution. This shows that $\mathscr{X}$ is a vector space. What we have shown above is that the set of $2n$ number of matrix functions, $x_{mk_c}(t) \triangleq \bar{x}_{mk} \cos(\omega_k t)$ and $x_{mk_s}(t) \triangleq \bar{x}_{mk} \cos(\omega_k t + \pi/2) = \bar{x}_k \sin(\omega_k t)$ for $i = 1, 2, \cdots, n$, form a basis for $\mathscr{X}$ and hence that any general solution can be expressed as a linear combination of these basis solutions.

We will now show that there exists a co-ordinate transformation in which the coupled set of n second order ODEs reduce to a set of uncoupled second order ODEs. Let T be the modal matrix of Ω. That is, the columns of T are the eigenvectors of Ω. Then it follows that $\Omega T = TD$ where $D = \text{diag}\{\omega_1^2, \omega_2^2, \cdots, \omega_n^2\}$ and hence that $D = T^{-1}\Omega T$. Consider the linear co-ordinate transformation $x = Tz$. Then we find that in these new co-ordinates the equations (4.54) become $\ddot{z} + Dz = 0$. Writing it out explicitly we obtain the n uncoupled second order ODEs

$$\begin{aligned}
\ddot{z}_1 + \omega_1^2 z_1 &= 0,\\
\ddot{z}_2 + \omega_2^2 z_2 &= 0,\\
&\vdots\\
\ddot{z}_n + \omega_n^2 z_n &= 0.
\end{aligned}$$

Each of these represent a particular mode of free vibration of the system and corresponds to a simple spring-mass system. Notice that what this says is that the unforced coupled system (4.54) can in fact be visualized as a system of uncoupled spring-mass systems in the new transformed coordinates $z = T^{-1}x$. We know that the general solution of each of these uncoupled systems take the form:

$$z_k(t) = \alpha_i \cos(\omega_k t + \phi_k)$$

for some unknowns $\{\alpha_k, \phi_k\}$ that will depend on the initial conditions. Then using the fact that $x = Tz$ we can write the solution of the system in the original coordinates as

$$x(t) = \sum_{k=1}^{n} z_k(t)\,\bar{x}_{mk} = \sum_{k=1}^{n} \alpha_k\,\bar{x}_{mk}\cos(\omega_k t + \phi_k),$$

demonstrating again how a general solution is a linear combination of the modal solutions. Notice that when $x(0) \approx \bar{x}_{mi}$ and $\dot{x}(0) = 0$ then

$$x(t) \approx \bar{x}_{mk} z_k(t) = \bar{x}_{mk}\cos\omega_k t.$$

4.3.2 Forced Vibration and Resonance

In the new coordinates $z = T^{-1}x$ we see that (4.53) takes the form

$$\ddot{z} + Dz = T^{-1}M^{-1}f(t). \tag{4.58}$$

Consider the case where one or all the degrees of freedom of the original system (4.53) are excited by a force of the form $f_0\cos\omega t$ where f_0 is some constant. Then $f(t)$ is the form $\bar{f}\cos\omega t$ where $\bar{f}$ is a constant $n \times 1$ matrix. Let $T^{-1}M^{-1}\bar{f} = \bar{u} = [\bar{u}_1 \;\; \bar{u}_2 \;\; \cdots \bar{u}_n]^T$ and then equation (4.58) takes the explicit form

$$\begin{aligned}
\ddot{z}_1 + \omega_1^2 z_1 &= \bar{u}_1\cos\omega t,\\
\ddot{z}_2 + \omega_2^2 z_2 &= \bar{u}_2\cos\omega t,
\end{aligned}$$

$$\vdots$$
$$\ddot{z}_n + \omega_n^2 z_n = \bar{u}_n \cos\omega t.$$

Thus we see that each of the modes are also excited by a sinusoidal force of the form $\bar{u}_i \cos\omega t$. Thus we see that the steady state solutions (zero initial condition solutions) takes the form

$$z_k(t) = \chi_k(\omega)\cos(\omega t + \phi_k(\omega)),$$

Here $\chi_k(\omega) = |G_k(\omega)|$ and $\phi_k(\omega) = \angle G_k(\omega)$ where

$$G_k(\omega) = \frac{1}{-\omega^2 + \omega_k^2}.$$

This shows us that if the forcing frequency is close to ω_k, the frequency corresponding to the k^{th} mode, then only $z_k(t)$ will have a very large amplitude compared to the rest of the $z_j(t)$. Therefore we can conclude that when the forcing frequency is close to any of the frequencies corresponding to a mode we will observe a large amplitude motion of that mode. Since $x = Tz$ we see that

$$x(t) = \sum_{k=1}^{n} z_k(t)\bar{x}_{mk} = \sum_{k=1}^{n} \chi_k(\omega)\cos(\omega t + \phi_k(\omega))\bar{x}_{mk},$$

and hence that when the forcing frequency ω is close to ω_k the steady state response of the system $x(t) \approx \chi_k(\omega)\cos(\omega t + \phi_k(\omega))\bar{x}_{mk}$ and hence takes the shape of the k^{th} mode $\bar{x}_{mk}$.

4.3.3 Example on Modes of Atomic Vibrations

In this section we provide an example of how one may approximate the atomic vibrations in a solid using classical Newtonian mechanics[1]. We consider the 1-D, poly atomic, N-periodic, lattice with nearest neighbor interactions. The cells of the lattice are indexed by $n = 1, \cdots, N$ while the poly atoms within a cell are indexed by $\alpha = 1, \cdots, s$. Denote the displacement of the α^{th} poly atom of the n^{th} unit cell by $q(\alpha, n)$. The periodic boundary conditions imply $q(\alpha, n+N) = q(\alpha, n)$ and $q(\alpha + rs, n) = q(\alpha, n+r)$ for all α and n. If $U(r(\alpha, n))$ is the interaction potential between the $(\alpha+n)^{\text{th}}$ and $(\alpha - 1 + n)^{\text{th}}$ particle, where $r(\alpha, n) = q(\alpha, n) - q(\alpha - 1, n)$, the equations of motion of the lattice are given by

$$m_\alpha \ddot{q}(\alpha, n) = -U'(q(\alpha, n) - q(\alpha - 1, n)) + U'(q(\alpha + 1, n) - q(\alpha, n)), \tag{4.59}$$

where an overdot means differentiation with respect to time and an overprime means differentiating with respect to the position variable. A crucial assumption here is that the inter-atomic potential between each particle is identical.

In the case of the harmonic lattice $U(r(\alpha, n)) = \frac{b_\alpha}{2} r(\alpha, n)^2$ and we have

[1] This is supplementary reading.

$$m_\alpha \ddot{q}(\alpha,n) = \left(b_{\alpha+1}q(\alpha+1,n) - (b_\alpha + b_{\alpha+1})q(\alpha,n) + b_\alpha q(\alpha-1,n)\right). \tag{4.60}$$

Functions $\psi_k(\alpha,n) = \frac{1}{\sqrt{m_\alpha N}}\phi_k(\alpha)e^{ikasn}$, where $\phi_k(\alpha)$ satisfy the periodicity conditions $\phi_k(\alpha+s) = e^{ikas}\phi_k(\alpha)$, satisfy the boundary conditions of the N-periodic poly atomic lattice. The stationary lattice vibrations of the form

$$q_k(\alpha,n;t) = c_k e^{-i\omega t}\psi_k(\alpha,n),$$

are a solution of the lattice equations if and only if ω satisfies the following difference equation:

$$\omega^2\phi_k(\alpha) = \left(-\frac{b_\alpha}{\sqrt{m_\alpha m_{\alpha-1}}}z^{-1} + \frac{(b_\alpha + b_{\alpha+1})}{\sqrt{m_\alpha m_\alpha}} - \frac{b_{\alpha+1}}{\sqrt{m_\alpha m_{\alpha+1}}}z\right)\phi_k(\alpha), \tag{4.61}$$

along with the periodic boundary conditions $\phi_k(\alpha+s) = e^{ikas}\phi_k(\alpha)$ where z^p is the p^{th} shift operator acting on the discrete variable α.

This is true if and only if

$$\left|\omega^2 I - A(k)\right| = 0, \tag{4.62}$$

where $A(k)$ is the complex s-periodic Jacobi matrix given by

$$A(k) = \begin{bmatrix} \frac{(b_1+b_2)}{\sqrt{m_1m_1}} & -\frac{b_2}{\sqrt{m_1m_2}} & 0 & \cdots & 0 & 0 & -e^{-ikas}\frac{b_1}{\sqrt{m_1m_s}} \\ -\frac{b_2}{\sqrt{m_2m_1}} & \frac{(b_2+b_3)}{\sqrt{m_2m_2}} & -\frac{b_3}{\sqrt{m_2m_3}} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{b_3}{\sqrt{m_3m_2}} & \frac{(b_3+b_4)}{\sqrt{m_3m_3}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{b_{s-2}}{\sqrt{m_{s-2}m_{s-3}}} & \frac{(b_{s-2}b_{s-1})}{\sqrt{m_{s-2}m_{s-2}}} & -\frac{b_{s-1}}{\sqrt{m_{s-2}m_{s-1}}} \\ 0 & 0 & 0 & \cdots & -\frac{b_{s-1}}{\sqrt{m_{s-1}m_{s-2}}} & \frac{(b_{s-1}+b_s)}{\sqrt{m_{s-1}m_{s-1}}} & -\frac{b_s}{\sqrt{m_{s-1}m_s}} \\ -e^{ikas}\frac{b_1}{\sqrt{m_sm_1}} & 0 & 0 & \cdots & 0 & -\frac{b_s}{\sqrt{m_sm_{s-1}}} & \frac{(b_s+b_1)}{\sqrt{m_sm_s}} \end{bmatrix},$$

and thus ω^2 is an eigenvalue of $A(k)$ and for each k there exists s eigenvalues given by

$$\omega_{\beta k} = \omega_\beta(k), \;\; \text{for } \beta = 1,\cdots,s.$$

These s number of expressions relating the wave vector k to the frequency of stationary vibrations are referred to as the s branches of the dispersion curve. The normalized eigenmode of the lattice associated with the frequency $\omega_{\beta k}$ is

$$\psi_{\beta k}(\alpha,n) = \frac{1}{\sqrt{m_\alpha N}}\phi_{\beta k}(\alpha)e^{ikasn},$$

where $\phi_{\beta k}(\cdot)$ is the normalized eigenvector of $A(k)$ associated with the eigenvalue $\omega_\beta^2(k)$. It can be shown, using properties of periodic Jacobi matrices, that $\phi_{\beta k}(\alpha+s) = e^{ikas}\phi_{\beta k}(\alpha)$ and that the eigenvectors of $A(k)$ are orthogonal. Thus $\psi_{\beta k}(\alpha,n)$ form an orthonormal basis for lattice displacements and a general lattice vibration can be expressed as

$$q(\alpha,n;t) = \sum_\beta \sum_k Q_{\beta k}(t)\psi_{\beta k}(\alpha,n). \tag{4.63}$$

Pre-multiplying (4.61) by $\phi_{\beta' k}(\alpha)$ and summing over α it can be shown that

$$\omega_{\beta k}^2 \delta(\beta-\beta') = \sum_{\alpha} b_\alpha \rho_{\beta'(-k)}(\alpha)\rho_{\beta k}(\alpha), \tag{4.64}$$

where

$$\rho_{\beta k}(\alpha) = \left(\frac{\phi_{\beta k}(\alpha)}{\sqrt{m_\alpha}} - \frac{\phi_{\beta k}(\alpha-1)}{\sqrt{m_{\alpha-1}}}\right).$$

Using (4.63) it follows that the kinetic energy of the lattice is given by

$$\mathrm{KE} = \sum_{\beta,k}\frac{1}{2}\dot{Q}_{\beta k}^*(t)\dot{Q}_{\beta k}(t) = \sum_{\beta,k}\frac{1}{2}|\dot{Q}_{\beta k}(t)|^2.$$

Observing that

$$q(\alpha,n;t) - q((\alpha-1),n;t) = \sum_{\beta,k}\frac{1}{\sqrt{N}}\rho_{\beta k}(\alpha)Q_{\beta k}(t)e^{ikasn}, \tag{4.65}$$

and using (4.64) it can be shown that the potential energy is given by

$$\mathrm{PE} = \sum_{\beta,k}\frac{1}{2}\left(\sum_{\alpha} b_\alpha \rho_{\beta(-k)}(\alpha)\rho_{\beta k}(\alpha)\right)Q_{\beta k}^*(t)Q_{\beta k}(t) = \sum_{\beta,k}\frac{\omega_{\beta k}^2}{2}|Q_{\beta k}(t)|^2.$$

The $Q_{\beta k}(t)$ are referred to as the complex normal co-ordinates. The requirement that the displacement $q(\alpha,n;t)$ be real implies that $Q_{\beta k}^* = Q_{\beta(-k)}$. The generalized momentum conjugate to $Q_{\beta k}$ is $P_{\beta k} = \dot{Q}_{\beta k}^*$.

The canonical transformations given by

$$Q_{\beta k} = \frac{1}{2}\left(z_{\beta k} + z_{\beta -k} + \frac{i}{\omega_{\beta k}}\left(p_{\beta k} - p_{\beta -k}\right)\right), \tag{4.66}$$

$$P_{\beta k} = \frac{1}{2}\left(p_{\beta k} + p_{\beta -k} - i\omega_{\beta k}\left(z_{\beta k} - z_{\beta -k}\right)\right). \tag{4.67}$$

relates the real normal co-ordinates $z_{\beta k}$ and its conjugate momentum $p_{\beta k}$ to their complex counterparts. Using the real normal co-ordinates the classical Hamiltonian takes the uncoupled form:

$$H_0(z) = \sum_{\beta,k}\frac{1}{2}\left(p_{\beta k}^2 + \omega_{\beta k}^2 z_{\beta k}^2\right), \tag{4.68}$$

and the corresponding classical Hamiltonian equations take the uncoupled form,

$$\ddot{z}_{\beta k} = -\omega_{\beta k}^2 z_{\beta k}. \tag{4.69}$$

In the semi-classical quantum treatment these oscillators are considered to be quantum oscillators.

Setting $s = 1$ and $\phi_k(1) = 1$ we recover the monatomic 1D lattice. Note that $\phi_k(0) = \phi_k(1-1) = e^{-ika}\phi(1)$.

Fig. 4.14 Axially loaded beam

4.4 Free Vibration of an Axially Loaded Beam

Consider the axially loaded beam in the applied mechanics lab. A schematic of the setup is shown in Figure: 4.14. It can be shown that, if the damping in the system is negligible and the displacements are small then the motion of the system is sufficiently accurately described by the solutions of the partial differential equation given by

$$\frac{\partial^2 y}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4}+\frac{P}{\rho A}\frac{\partial^2 y}{\partial x^2}=0 \tag{4.70}$$

where y is the transverse displacement of the beam, P is the axial compressive force, ρ is the density of the material of the beam, and E is the modulus of rigidity, I is the cross sectional moment of inertia, and A is the cross sectional area of the uniform beam. In this section we will consider the case where the axial load is a constant.

When we observe any particular point on the beam when it exhibits a vibratory motion, we see that the point oscillates periodically. Thus it is reasonable to search for solutions of the form

$$y(x,t)=\psi(x)\cos(\omega t+\phi). \tag{4.71}$$

If such solutions exist then they should satisfy the above PDE and the boundary conditions

$$y(0,t)=0, \quad y(L,t)=0, \tag{4.72}$$

$$\frac{\partial^2 y}{\partial x^2}(0,t)=0, \quad \frac{\partial^2 y}{\partial x^2}(L,t)=0, \tag{4.73}$$

where L is the length of the beam. Let us proceed to check if there exists a solution of the form (4.71) such that it satisfies the PDE and the boundary conditions given by (4.72)–(4.73). Substituting (4.71) in the PDE we have

$$\left(-\omega^2\psi + \frac{EI}{\rho A}\frac{d^4\psi}{dx^4} + \frac{P}{\rho A}\frac{d^2\psi}{dx^2}\right)\cos(\omega t + \phi) = 0.$$

Since this expression should be true for all t and $\cos\omega t$ is not identically zero we have that for the existence of a solution of the form (4.71) the function $\psi(x)$ should satisfy

$$\frac{EI}{\rho A}\frac{d^4\psi}{dx^4} + \frac{P}{\rho A}\frac{d^2\psi}{dx^2} - \omega^2\psi = 0. \tag{4.74}$$

The requirement that the boundary conditions (4.72)–(4.73) be satisfied imply that $\psi(x)$ should satisfy

$$\psi(0) = 0, \quad \psi(L) = 0, \tag{4.75}$$

$$\frac{d^2\psi}{dx^2}(0) = 0, \quad \frac{d^2\psi}{dx^2}(L) = 0. \tag{4.76}$$

Thus the problem of finding a solution of the form (4.71 reduces to that of solving a two-point boundary value problem. That is to a problem of solving the ODE (4.74) subjected to the boundary conditions (4.75)–(4.76).

We notice that functions of the form

$$\psi_k(x) = \sin\left(\frac{k\pi x}{L}\right) \tag{4.77}$$

for any integer k satisfies the boundary conditions (4.75)–(4.76). Thus it is reasonable to try to see if $\psi_k(x)$ satisfies the ODE (4.74). The function (4.77) satisfies the ODE (4.74) if and only if

$$\left(\frac{EI}{\rho A}\frac{k^4\pi^4}{L^4} - \frac{P}{\rho A}\frac{k^2\pi^2}{L^2} - \omega^2\right)\sin\left(\frac{k\pi x}{L}\right) = 0.$$

Thus in the case where $P = P_o$ is a constant, non-trivial $\psi_k(x)$ that satisfies the ODE (4.74) exist if and only if ω satisfies

$$\omega^2 = \omega_k^2 = \frac{k^2\pi^2}{\rho A L^2}\left(\frac{EIk^2\pi^2}{L^2} - P_o\right). \tag{4.78}$$

Therefore we see that each

$$y_k(x,t) = \psi_k(x)\cos(\omega_k t + \phi_k) = \sin\left(\frac{k\pi x}{L}\right)\cos(\omega_k t + \phi_k) \tag{4.79}$$

satisfies the PDE and the boundary conditions for any $k = 1, 2, \cdots$. These are called to modes of vibration of the beam.

Since the PDE is linear we see that any linear combination of the modes is also a solution. Thus

$$y(x,t) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(\omega_k t + \phi_k\right) \tag{4.80}$$

is also a solution where $\alpha_1, \alpha_2, \cdots$ and $\phi_1, \phi_2, \cdots$ are unknown arbitrary constants. The question that we need to answer is if any given solution can be written in the above form. It can be shown that for this class of PDEs the initial conditions $y(x,0) = y_0(x)$ and $\frac{\partial y}{\partial t}(x,0) = v_0(x)$ and the boundary conditions uniquely determine the solution. Since the above form satisfies the boundary conditions what we need to see is if we can uniquely determine the $\alpha_1, \alpha_2, \cdots$ and $\phi_1, \phi_2, \cdots$ such that $y(x,0) = y_0(x)$ and $\frac{\partial y}{\partial t}(x,0) = v_0(x)$ is satisfied. That is we need to answer if we can uniquely determine the $\alpha_1, \alpha_2, \cdots$ and $\phi_1, \phi_2, \cdots$ such that

$$y_0(x) = \sum_{k=1}^{\infty} \alpha_k \cos\phi_k \sin\left(\frac{k\pi x}{L}\right), \tag{4.81}$$

$$v_0(x) = \sum_{k=1}^{\infty} -\omega_k \alpha_k \sin\phi_k \sin\left(\frac{k\pi x}{L}\right), \tag{4.82}$$

for any given $y_0(x)$ and $v_0(x)$ that satisfies the boundary conditions. Let $\beta_k = \alpha_k \cos\phi_k$ and $\gamma_k = -\omega_k \alpha_k \sin\phi_k$. Then the above two equations can be written as

$$y_0(x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right), \tag{4.83}$$

$$v_0(x) = \sum_{k=1}^{\infty} \gamma_k \sin\left(\frac{k\pi x}{L}\right). \tag{4.84}$$

Thus we see that if we can find unique $\beta_1, \beta_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$ that satisfy the above two equations we would succeed in solving the problem of finding a solution to (4.70) that satisfies the boundary conditions and the given initial conditions.

Observe that

$$\frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}. \tag{4.85}$$

Multiplying (4.83) and (4.84) by $\frac{2}{L}\sin\left(\frac{l\pi x}{L}\right)$ and integrating over the interval $[0,L]$ we have

$$\frac{2}{L}\int_0^L y_0(x) \sin\left(\frac{l\pi x}{L}\right) dx = \sum_{k=1}^{\infty} \beta_k \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi x}{L}\right) dx, \tag{4.86}$$

$$\frac{2}{L}\int_0^L v_0(x) \sin\left(\frac{l\pi x}{L}\right) dx = \sum_{k=1}^{\infty} \gamma_k \frac{2}{L}\int_0^L \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{l\pi x}{L}\right) dx. \tag{4.87}$$

Now from (4.85) we have that

$$\beta_l = \frac{2}{L}\int_0^L y_0(x)\sin\left(\frac{l\pi x}{L}\right)dx, \tag{4.88}$$
$$\gamma_l = \frac{2}{L}\int_0^L v_0(x)\sin\left(\frac{l\pi x}{L}\right)dx, \tag{4.89}$$

and therefore we have found a solution to (4.70) that satisfies the boundary conditions and the given initial conditions. It can be shown that the solution to this type of problem is unique. Thus we see that any general solution of (4.70) can be uniquely expressed as (4.80).

Note that the space $\mathscr{F}$ of sufficiently smooth functions defined on the interval $[0,L]$ that satisfy the boundary conditions is an infinite dimensional vector space. Then the relationship

$$\langle f,g\rangle \triangleq \frac{2}{L}\int_0^L f(x)g(x)\,dx \tag{4.90}$$

for $f,g\in\mathscr{F}$ defines an inner product on $\mathscr{F}$. With respect to this inner product we have that the functions $\psi_k = \sin\left(\frac{k\pi x}{L}\right)$, are orthonormal to each other. That is

$$\langle\psi_k,\psi_l\rangle = \begin{cases} 0 & \text{if } k\neq l \\ 1 & \text{if } k = l \end{cases}$$

Using these orthonormal functions we can write (4.83) and (4.84) as

$$y_0(x) = \sum_{k=1}^{\infty}\beta_k\psi_k(x), \tag{4.91}$$
$$v_0(x) = \sum_{k=1}^{\infty}\gamma_k\psi_k(x). \tag{4.92}$$

Thus we see that

$$\langle\psi_l,y_0\rangle = \sum_{k=1}^{\infty}\beta_k\langle\psi_l,\psi_k\rangle,$$
$$\langle\psi_l,v_0\rangle = \sum_{k=1}^{\infty}\gamma_k\langle\psi_l,\psi_k\rangle.$$

Which gives us

$$\beta_l = \langle\psi_l,y_0\rangle,$$
$$\gamma_l = \langle\psi_l,v_0\rangle.$$

Compare this expression with expressions for β_l and γ_l for the case where the system is of the form $\ddot{x}+\Omega x = 0$.

On a side note: notice that this says that any function $y_0\in\mathscr{F}$ can be uniquely written down as a linear combination of the orthonormal vectors $\psi_1,\psi_2,\cdots$. Thus we see that the set of vectors $\psi_1,\psi_2,\cdots$ form an orthonormal basis for $\mathscr{F}$. We say that (4.91) is the *Fourier*

Series of $y_0(x)$ and the $\beta_1, \beta_2, \cdots$, the *Fourier Co-efficients*. Similarly We say that (4.92) is the Fourier Series of $v_0(x)$ and the $\gamma_1, \gamma_2, \cdots$, the Fourier Co-efficients.

In what follows, I would like to demonstrate that this process that we have followed is very general. Observe that (4.70) can be expressed as

$$\frac{\partial^2 y}{\partial t^2} + Hy = 0 \tag{4.93}$$

where $H : \mathscr{F} \to \mathscr{F}$ given by

$$H = \frac{EI}{\rho A}\frac{\partial^4}{\partial x^4} + \frac{P_o}{\rho A}\frac{\partial^2}{\partial x^2}, \tag{4.94}$$

is a linear operator on functions defined on the interval $[0, L]$ that satisfy the boundary conditions. Let us consider this class of problems and try to see if they admit solutions of the form

$$y(x,t) = \psi(x)\cos(\omega t + \phi), \tag{4.95}$$

where $\psi \in \mathscr{F}$. Substituting this in (4.93) we have the $y(x,t) = \psi(x)\cos(\omega t + \phi)$ is a solution of (4.93) if and only if

$$(-\omega^2 + H)\psi(x)\cos(\omega t + \phi) = 0. \tag{4.96}$$

Since $\cos(\omega t + \phi) \neq 0$ we see that the above is true if and only if there exists $\psi(x) \neq 0$ such that

$$(-\omega^2 + H)\psi(x) = 0. \tag{4.97}$$

Notice that this is an eigenvalue problem. In the case of the axially loaded beam, H is given by (4.94). Thus in this case we see that the functions of the form $\sin\left(\frac{k\pi x + \varphi_k}{L}\right)$ are eigenfunctions of H with corresponding eigenalues ω_k^2 given by (4.78) for any $k \in \mathbb{R}$. Notice that there are infinitely many such continuum number of eigenfunctions. However we observe that not all of these satisfy the given boundary conditions. Only the functions that satisfy $\varphi_k = 0$ and $k = 1, 2, \cdots$ satisfy the boundary conditions. Thus when H is restricted to $\mathscr{F}$ the only eigenfunctions are $\psi_k(x) = \sin\left(\frac{k\pi x}{L}\right)$ with eigenvalues given by (4.78) for any integer $k = 1, 2, \cdots$. Thus we see that there exists infinitely many discrete number of solutions $y_k(x,t) = \psi_k(x)\cos(\omega_k t + \phi_k)$ that satisfy the PDE (4.93) and the boundary conditions. Since the PDE (4.93) is linear any linear combination of these eigensolutions are also a solution of (4.93). Thus when the eigensolutions are discretely many then any general solution can be written down as

$$y(x,t) = \sum_{k=1}^{\infty} \alpha_k \psi_k(x)\cos(\omega_k t + \phi_k). \tag{4.98}$$

In the case of the axially loaded beam with pinned ends we have shown that the eigenfunctions $\psi_1(x), \psi_2(x), \cdots$ are orthonormal with respect to the inner product

$$\langle f, g \rangle \triangleq \frac{2}{L}\int_0^L f(x)g(x)\,dx \tag{4.99}$$

defined on $\mathscr{F}$. Using the inner product we find that

$$\alpha_l \cos\phi_l = \langle \psi_l, y(x,0)\rangle,$$
$$-\omega_l \alpha_l \sin\phi_l = \left\langle \psi_l, \frac{\partial y}{\partial t}(x,0)\right\rangle,$$

uniquely determines the α_k and ϕ_k.

In general if the linear operator satisfies certain conditions it can be shown, similarly to the case of the axially loaded beam, that there exists infinitely many such eigensolutions. The boundary conditions will determine if the eigenvalues are discrete or continuous. In most cases one can show that there exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{F}$ such that the set of eigensolutions are orthonormal. Thus as before one can show that any given solution of (4.93) can be expressed as a linear combination of these eigensolutions. If the eigenfunctions are discrete this linear combination takes the form of an infinite series while if the eigenfunctions are continuously many then the linear combination takes the form of an integral. This idea of expressing any any solution using a sum of basis solutions is the notion of a Fourier series. Thus we make a slight digression to study it in the next section.

4.4.1 The Basic Idea of Fourier Series

Consider the vector space of square integrable complex valued functions defined on the finite interval $[0,L]$. Denote this space by $\mathscr{F}$. That is if $f \in \mathscr{F}$ then $f : [0,L] \to \mathbb{C}$ and

$$\int_0^L f^*(x)f(x)\,dx < \infty.$$

For $f, g \in \mathscr{F}$ one can easily show that

$$\langle f, g \rangle \triangleq \frac{1}{L}\int_0^L f^*(x)g(x)\,dx \tag{4.100}$$

defines an inner product on the vector space $\mathscr{F}$. It is easy to show that the set of functions

$$\psi_k(x) = e^{i\frac{2k\pi x}{L}} \tag{4.101}$$

for any $k = 0, \pm 1, \pm 2, \cdots$, are orthonormal to each other with respect to this inner product. The question is do they span $\mathscr{F}$. To find the answer we seek to see if any $f \in \mathscr{F}$ can be written as a linear combination of the $\psi_k(x)$. That is given a $f \in \mathscr{F}$ can we uniquely determine the coefficients $\alpha_k \in \mathbb{C}$ such that

$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_k \psi_k(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{i\frac{2k\pi x}{L}}. \tag{4.102}$$

Using the inner product one can show that

$$\alpha_k = \langle \psi_k, f \rangle = \frac{1}{L}\int_0^L f(x)e^{-i\frac{2k\pi x}{L}}\,dx \tag{4.103}$$

Thus any $f \in \mathscr{F}$ can be uniquely expressed as (4.102) provided the infinite series (4.102) converges. It can be shown that this series in fact converges. This is called the *Fourier Series* of $f \in \mathscr{F}$.

In many practical situations of interest, especially when dealing with electromagnetic signals, the functions of interest are square integrable functions defined on the interval $[0,\infty)$. Let us see how the above notions generalize to this case. That is let us investigate (4.102) and (4.103) when $\lim L \to \infty$. Define $\omega_k = \frac{2k\pi}{L}$. Then one can define $\delta\omega_k = \frac{2\pi}{L}$. Then one can re-write (4.102) as

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} L\alpha_k e^{i\omega_k x}\delta\omega_k.$$

$$F(\omega_k) = L\alpha_k = \int_0^L f(x)e^{-i\omega_k x}\,dx.$$

Thus in the limit $\lim L \to \infty$ the above expressions become

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega x}\,d\omega. \tag{4.104}$$

$$F(\omega) = \int_0^{\infty} f(x)e^{-i\omega x}\,dx. \tag{4.105}$$

In the case where we deal with square integrable functions the Fourier coefficients (4.105) are called the *Fourier Transform* of $f(x)$ and the Fourier integral (4.104) is called the *Inverse Fourier Transform* of $F(\omega)$.

4.5 Forced Vibration of an Axially Loaded Beam

In this section we will consider two types of periodic forcing acting on an axially loaded beam. In the first instance we will consider the case where the axially loaded beam is transversely excited using a periodic force distributed along the length of the beam and in the second instance we will consider the case where the axial load itself is periodically varying.

4.5.1 Transversely Forced Axially Loaded Beam

Consider the transversely forced axially loaded beam model written as

$$\frac{\partial^2 y}{\partial t^2} + Hy = q(x,t), \tag{4.106}$$

where $q(x,t) \in \mathscr{F}$ for all t. Since $q(x,t) \in \mathscr{F}$ for all t we can write

$$q(x,t) = \sum_{k=1}^{\infty} f_k(t)\psi_k(x),$$

where ψ_k are eigenvectors of H that satisfy the boundary conditions.

$$f_k(t) = \langle \psi_k, q(x,t) \rangle.$$

If the solutions exists we know that the solution $y(x,t) \in \mathscr{F}$ for all t as well. Thus the solution can be expressed as

$$y(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\psi_k(x).$$

Substituting this in the PDE we have

$$\sum_{k=1}^{\infty} (\ddot{\alpha}_k + \omega_k^2 \alpha_k - f_k)\psi_k = 0.$$

Since the ψ_k are linearly independent we have that necessarily

$$\ddot{\alpha}_k + \omega_k^2 \alpha_k = f_k,$$

where ω_k^2 is the eigenvalue of H corresponding to the eigenvector ψ_k. Since this is a linear system one can easily solve this. In the case where the initial conditions are zero and the forcing is of the form $q(x,t) = q_s(x)\cos\omega t$ we see that

$$\alpha_k(t) = \beta_k \chi_k(\omega)\cos(\omega t + \phi_k(\omega)),$$

where $\beta_k = \langle \psi_k, q_s \rangle$.

Here $\chi_k(\omega) = |G_k(\omega)|$ and $\phi_k(\omega) = \angle G_k(\omega)$ where

$$G_k(\omega) = \frac{1}{-\omega^2 + \omega_k^2}.$$

Thus in the case where the forcing is of the form $q(x,t) = q_s(x)\cos\omega t$ and the initial conditions are zero the forced vibration of the axially loaded beam can be explicitly written down as

$$y(x,t) = \sum_{k=1}^{\infty} \langle \psi_k, q_s \rangle \chi_k(\omega)\cos(\omega t + \phi_k(\omega)) \sin\left(\frac{k\pi x}{L}\right).$$

Observe that resonance occurs when $\omega = \omega_k$ for each k.

4.5.2 Periodically Varying Axially Loaded Beam

Let us consider the axially loaded beam (4.70) and investigate what happens if the axial load is of the form $P(t) = P_o - f(t)$ where $f(t)$ is periodic in time with period T and P_o is constant. Then the (4.70) beam takes the form

$$\frac{\partial^2 y}{\partial t^2} + Hy - \frac{f(t)}{\rho A}\frac{\partial^2 y}{\partial x^2} = 0, \tag{4.107}$$

where

$$H = \frac{EI}{\rho A}\frac{\partial^4}{\partial x^4} + \frac{P_o}{\rho A}\frac{\partial^2}{\partial x^2},$$

We have seen that the eigen functions $\psi_k(x) = \sin\left(\frac{k\pi x}{L}\right)$ of H with corresponding eigenvalues ω_k^2 given by (4.78) are a basis for the solution space. Thus as done in the previous section let us see if the system admits solutions of the form $y_k(x,t) = \gamma_k(t)\sin\left(\frac{k\pi x}{L}\right)$. Substituting this in the PDE we have

$$\left(\ddot{\gamma}_k + \left(\omega_k^2 + \frac{f(t)}{\rho A}\frac{k^2\pi^2}{L^2}\right)\gamma_k\right)\sin\left(\frac{k\pi x}{L}\right) = 0.$$

Thus the $\gamma_k(t)$ should satisfy

$$\frac{d^2\gamma_k}{dt^2} + (\omega_k^2 + \mu_k f(t))\gamma_k = 0, \tag{4.108}$$

where

$$\omega_k^2 = \frac{k^2\pi^2}{\rho A L^2}\left(\frac{EIk^2\pi^2}{L^2} - P_o\right)$$

and $\mu_k = \frac{k^2\pi^2}{\rho A L^2}$. Thus the solution of the axially loaded beam (4.70) will be of the form

$$y(x,t) = \sum_{k=1}^{\infty}\gamma_k(t)\sin\left(\frac{k\pi x}{L}\right), \tag{4.109}$$

where each $\gamma_k(t)$ are the solutions of the equation (4.108) with initial conditions $\gamma_k(0) = \langle\sin\left(\frac{k\pi x}{L}\right), y_0\rangle$ and $\dot{\alpha}_k(0) = \langle\sin\left(\frac{k\pi x}{L}\right), v_0\rangle$.

The above solution is stable as long as all the $\gamma_k(t)$ are stable. Thus it is of practical interest to investigate the solutions of the equation (4.108) and stability of the origin when the axial forcing $f(t)$ is periodic. That is when $f(t+T) = f(t)$ for all t. This equation is called the Hill's equation and in the special case where $f(t)$ is sinusoidal it is called the Mathieu equation.

We note that the Hill's equation (4.108) can be written down as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\omega^2 + \mu f(t)) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{4.110}$$

where $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. That is it takes the form $\dot{x} = A(t)x$ where $A(t)$ is periodic. We will study this general case below.

4.6 Floquet Theory

Consider the time varying linear system

$$\dot{x} = A(t)x, \tag{4.111}$$

where $A(t+T) = A(T)\ \forall\ t \in \mathbb{R}$. The study of the solutions of this type of systems is called Floquet theory.

Let $\Phi(t)$ be a fundamental matrix solution of this system. That is $\Phi(t)$ is a solution of the matrix differential equation

$$\dot{\Phi} = A(t)\Phi, \tag{4.112}$$

with initial condition $\Phi(0) = I_{3\times 3}$.

Let $\Psi(t) = \Phi(t+T)$. Then $\dot{\Psi}(t) = \dot{\Phi}(t+T) = A(t+T)\Phi(t+T) = A(t)\Phi(t+T) = A(t)\Psi(t)$ Thus we see that $\Phi(t+T)$ is also a solution of (4.112) with initial conditions $\Phi(T)$. Since $\Phi(t)$ is invertible for all t so is $\Psi(t) = \Phi(t+T)$. Thus $\Phi(t+T)$ is also a fundamental matrix solution of (4.112). One can show that $\Phi(t)\Phi(T)$ is also a fundamental matrix solution with initial condition $\Phi(T)$ thus from uniqueness of solutions we have that

$$\Phi(t+T) = \Phi(t)C \tag{4.113}$$

where we have defined $C \triangleq \Phi(T)$. Recall that $C = \Phi(T)$ is invertible. Thus there exists a $n\times n$ matrix ζ such that $C = e^{T\zeta}$. Then we see that the matrix function $Q(t) = \Phi(t)e^{-t\zeta}$ satisfies $Q(t+T) = \Phi(t+T)e^{-(t+T)\zeta} = \Phi(t+T)e^{-(t+T)\zeta} = \Phi(t)e^{T\zeta}e^{-(t+T)\zeta} = \Phi(t)e^{-t\zeta} = Q(t)$. Thus we see that

$$\Phi(t) = Q(t)e^{t\zeta}, \tag{4.114}$$

where $Q(t+T) = Q(t)$ for all t. Since $e^{T\zeta + i2\pi kI} = e^{i2\pi k}e^{T\zeta} = e^{T\zeta}$ we note that ζ is not unique.

Consider the coordinate transformation $x(t) = Q(t)y(t)$. Then

$$\dot{x} = Ax = \dot{Q}y + Q\dot{y} = \dot{\Phi}e^{-t\zeta}y - \Phi e^{-t\zeta}\zeta y + Q\dot{y}.$$

$$Ax = A\Phi e^{-t\zeta}y - \Phi e^{-t\zeta}\zeta y + Q\dot{y} = Ax - Q\zeta y + Q\dot{y}.$$

Which gives the time invariant system

$$\dot{y} = \zeta y. \tag{4.115}$$

the coordinate transformation $x(t) = Q(t)y(t)$ is known as the Lyapunov transformation. We state these results in the following theorem.

Theorem 4.1 (Floquet Theorem). *Consider the time varying periodic system $\dot{x} = A(t)x$ where $x(t) \in \mathbb{R}^n$ and $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Let $\Phi(t)$ be the solution of the matrix differential equation $\dot{\Phi} = A(t)\Phi$ with initial condition $\Phi(0) = I_{n\times n}$. We will call the matrix $\Phi(t)$ the fundamental matrix solution of the time varying periodic system. One can show the following properties of the fundamental matrix solution:*

(a) $\Phi(t)$ is defined for all $t \in \mathbb{R}$ and the inverse $\Phi^{-1}(t)$ exists for all $t \in \mathbb{R}$.
(b) $\Phi(t+T) = \Phi(t)\Phi(T)$.
(c) There exists an $n \times n$ matrix ζ such that $\Phi(T) = e^{T\zeta}$.
(d) There exists a matrix $Q(t)$ such that $Q(t+T) = Q(t)$ and $\Phi(t) = Q(t)e^{t\zeta}$.
(e) Let $y = Q(t)x$, then $\dot{y} = \zeta y$.

The eigenvalues of $\Phi(T)$, denoted by $\rho_1, \rho_2, \cdots, \rho_n$, are called the Floquet multipliers (characteristic multipliers) of the system and if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of ζ then they are called the characteristic exponents. Observe that they are related by $\rho_k = e^{\lambda_k T}$. The characteristic exponents are unique modulo $i2\pi k$ however the multipliers are unique.

Note that a solution $x(t)$ of (4.111) with initial condition $x(0)$ satisfies

$$x(t) = \Phi(t)x(0) = Q(t)e^{t\zeta}x(0). \tag{4.116}$$

We know that the equilibrium solution $x(t) \equiv 0$ is a solution. We are interested in knowing if solutions with initial conditions close to 0 will stay near 0, converge to 0, or diverge from 0. Since $Q(t)$ is periodic the above expression (4.116) shows that the answer to the above problem depends on the eigenvalues of ζ.

Let S be the invertible matrix such that $S^{-1}\zeta S = D + N$ where D is a diagonal matrix with eigenvalues of ζ in its diagonal and N a nilpotent matrix. Then we see that if $x(t) = Sz(t)$

$$z(t) = (S^{-1}Q(t)S)e^{tD}e^{tN}z(0). \tag{4.117}$$

Thus one can conclude the following:

Theorem 4.2 (Stability of Periodic Time Varying Linear Systems).

a. $\lim_{t\to\infty} x(t) \to 0$ *if and only if all eigenvalues of ζ are in the strict left half complex plane. That is if and only if $|\rho_k| < 1$.*
b. The system admits a periodic solution if ζ has a zero eigenvalue with multiplicity one. That is if there exists $\rho_k = 1$ with multiplicity one.
c. The system admits a 2-periodic solution if ζ has a eigenvalue of the form $\frac{i(2k+1)\pi}{T}$ with multiplicity one where k is any integer. That is if there exists $\rho_k = -1$ with multiplicity one.
d. The system admits solutions that become unbounded if ζ has eigenvalues in the strict right half plane. That is if there exists $|\rho_k| > 1$.

Since the above shows that the Floquet multipliers play a crucial role in determining the type of solutions the following property will become useful. Using the derivative of the det operator we have that

$$\rho_1\rho_2\cdots\rho_n = \det(\Phi(T)) = e^{\mathrm{trace}(T\zeta)} = e^{\int_0^T \mathrm{trace}(A(s))ds}. \tag{4.118}$$

The Case where $n = 2$.

In this case one finds that $e^{T(D+N)}$ must take one of the following forms.

$$e^{T(D+N)} = \begin{bmatrix} e^{\lambda_1 T} & 0 \\ 0 & e^{\lambda_2 T} \end{bmatrix}, \quad e^{T(D+N)} = \begin{bmatrix} e^{\lambda T} & 0 \\ 0 & e^{\lambda T} \end{bmatrix},$$

$$e^{T(D+N)} = \begin{bmatrix} e^{\lambda T} & Te^{\lambda T} \\ 0 & e^{\lambda T} \end{bmatrix}, \quad e^{T(D+N)} = e^{\sigma T} \begin{bmatrix} \cos \Omega T & -\sin \Omega T \\ \sin \Omega T & \cos \Omega T \end{bmatrix}.$$

In each case we have

$$\rho_1 \rho_2 = e^{\lambda_1 T} e^{\lambda_2 T}, \quad \rho_1 \rho_2 = e^{2\lambda T}, \quad \rho_1 \rho_2 = e^{2\lambda T}, \quad \rho_1 \rho_2 = e^{\sigma T}.$$

From these normal forms we can conclude the following:

For periodic time varying linear systems on $\mathbb{R}^2$ with period T,

(a.) for global asymptotic stability of the origin all the multipliers should be inside the unit disk.
(b.) if any of the multipliers are outside the unit disk then the origin is unstable.
(c.) if the multipliers are purely imaginary and of magnitude one then the origin is stable.
(d.) if the multipliers are equal to +1 and $e^{T(D+N)}$ takes the second form then the solutions are periodic with period T. The origin is unstable if $e^{T(D+N)}$ is of the third form.
(e.) if the multipliers are equal to -1 and $e^{T(D+N)}$ takes the second form then the solutions are periodic with period $2T$. The origin is unstable if $e^{T(D+N)}$ is of the third form.

4.7 The Mathieu Equation

We have seen how the problem of the stability of the modes of the periodically excited axially loaded beam reduces to that of the stability of the Mathieu equation in section-4.5.2. It has been found that the Mathieu equation also arises in electrical and thermal diffusion, electromagnetic wave guides, elliptical cylinders in viscous fluids, diffraction of sound and electromagnetic waves, elliptic membranes, ring antennas, alternating gradient focusing, the Paul trap for charged particles, the mirror trap for neutral particles, the stabilization of the inverted pendulum, and stabilization of electrostatic MEMS. In the sections below we will first apply the Floquet theory presented in the previous section to study the solutions of the Mathieu equation. Then we will these results to the study of the stability of the periodically excited axially loaded beam and the stabilization of the inverted pendulum.

First let us apply the Floquet theorem treated in the previous section to the study the stability of the origin of the normalized Mathieu equation

$$\ddot{y}+(\alpha+\varepsilon\cos t)y=0. \tag{4.119}$$

This equation can be expressed as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha+\varepsilon\cos t) & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4.120}$$

We are interested in answering the following questions about system (4.120). For what values of α and ε will

a. the origin be globally asymptotically stable?
b. periodic solutions of period $T=2\pi$ exist?
c. periodic solutions of period $T=4\pi$ exist?
d. the origin be stable?

Floquet theory tells us that the solution takes the forms

$$x(t)=P(t,\alpha,\varepsilon)e^{t\zeta(\alpha,\varepsilon)},$$

where $\zeta(\alpha,\varepsilon)$ is defined by

$$e^{T\zeta(\alpha,\varepsilon)}=\Phi(T,\alpha,\varepsilon),$$

with $T=2\pi$. We have explicitly indicated that P,ζ,Φ also depend on the parameters α and ε

We begin by noticing that ε is very small and that when $\varepsilon=0$ we have periodic solutions of period $2\pi/\sqrt{\alpha}$. In this case the state transition matrix is

$$\Phi(t,\alpha,0)=\begin{bmatrix} \cos\sqrt{\alpha}t & \frac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha}t \\ -\sqrt{\alpha}\sin\sqrt{\alpha}t & \cos\sqrt{\alpha}t \end{bmatrix}$$

Since $T=\frac{2\pi}{\sqrt{\alpha}}$ when $\varepsilon=0$, the characteristic multipliers are given by

$$\rho(T,\alpha,0)=\cos\sqrt{\alpha}T\pm i\sin\sqrt{\alpha}T.$$

Observe that $\rho(2\pi/\sqrt{\alpha},\alpha,0)$ are complex conjugate and are on the unit disk. From which we can conclude, as already we know, that periodic solutions of period $T=2\pi/\sqrt{\alpha}$ exists. We also see that when α is very small the multipliers are close to 1. The question we are interested in answering is what happens when $\varepsilon\neq 0$ but is very small.

From continuity arguments we could imagine that for very small ε the state transition matrix should also be close to $\Phi(t,0)$. Thus we assume

$$\Phi(t,\alpha,\varepsilon)=\Phi(t,\alpha,0)+\varepsilon\Phi_{\varepsilon}(t,\alpha)+\varepsilon^2\Phi_{\varepsilon^2}(t,\alpha)+\cdots$$

and

$$\rho(T,\alpha,\varepsilon)=\rho(T,\alpha,0)+\varepsilon\rho_{\varepsilon}+\varepsilon^2\rho_{\varepsilon^2}+\cdots$$

Thus for small epsilon it is plausible that the characteristic multipliers are also close to $\cos\sqrt{\alpha}T \pm i\sin\sqrt{\alpha}T$ as well.

From (4.118) we see that

$$\rho_1(T,\alpha,\varepsilon)\rho_2(T,\alpha,\varepsilon) = 1.$$

Thus for all α,ε the multipliers should either lie on the unit disk or be real. If they are real the multipliers are the reciprocal of each other. Thus if the multipliers are real one of the multipliers will necessarily be greater than one and hence the origin will be unstable. The question we need to answer is for what values of α,ε will the multipliers be purely imaginary or real. The following map that was generated numerically shows the regions in α,ε space where the multipliers are purely imaginary and are real.

The shaded regions corresponds to the case where the Floquet multipliers are on the unit disk (stable equilibrium). The unshaded area corresponds to the case where one Floquet multiplier is outside the unit disk (unstable equilibrium). The boundaries of the shaded regions correspond to the case where both Floquet multipliers are either +1 or -1.

Figure-4.16 (b) shows several typical locations on the stability map and Figure-4.16 (a) shows the corresponding location of the Floquet multipliers.

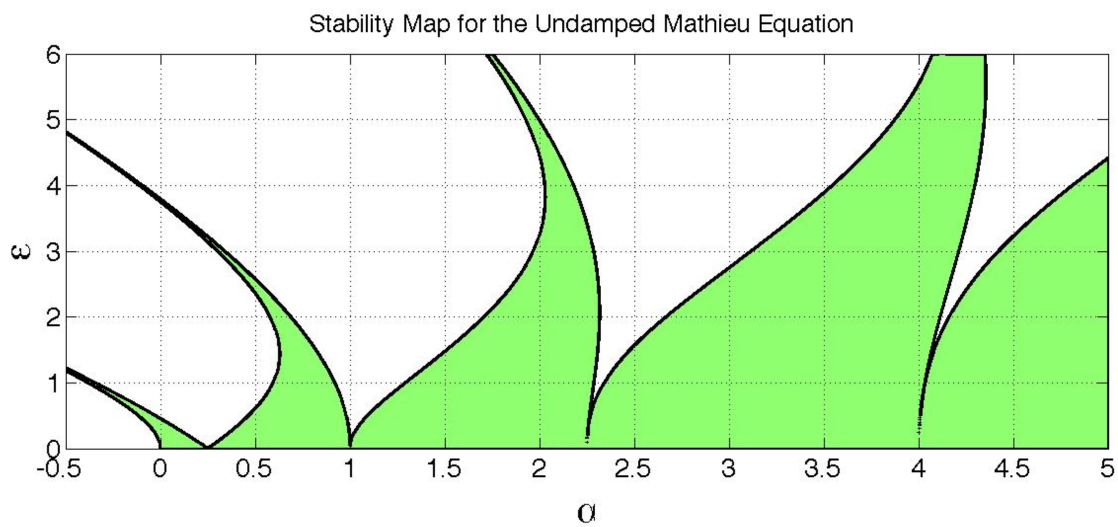
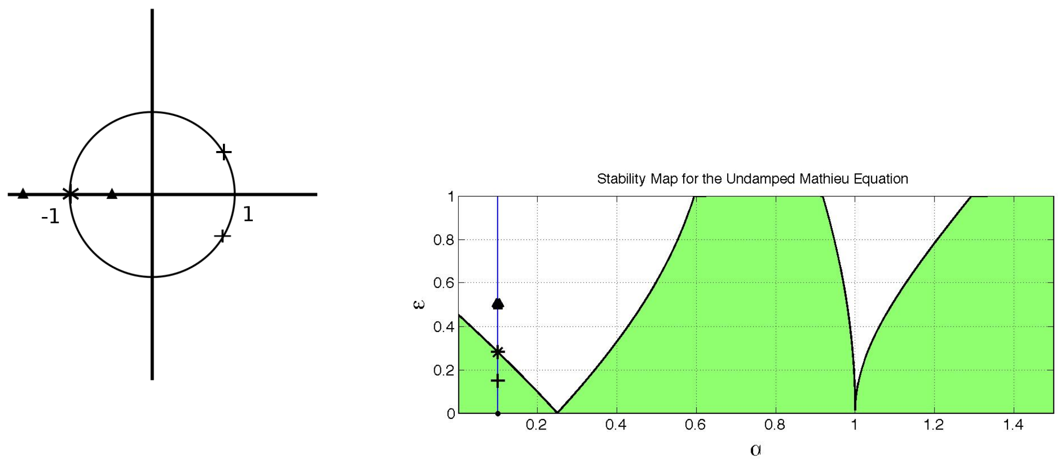


Fig. 4.15 Stability Map for the Mathieu Equation. Figures were created by J. M. Berg using a numerical algorithm in the text 'Nonlinear Ordinary Differential Equations' by Jordan and Smith.



(a) (b)

Fig. 4.16 Figure (a) shows the Locus of the Floquet Multipliers as ε varies for $\alpha = 0.1$. The corresponding points on the stability map are shown in figure (b)

4.7.1 Parametric Resonance of the Axially Loaded Beam

In this section we re-visit the periodically varying axially loaded beam considered in section-4.5.2 and investigate the stability of the solutions when the axial compressive load is periodic and is of the form $P(t) = P_o - a\cos\omega t$.

In section-4.5.2 we have seen that that solutions of the can be written as

$$y(x,t) = \sum_{k=1}^{\infty} \gamma_k(t) \sin\left(\frac{k\pi x}{L}\right),$$

where each $\gamma_k(t)$ are the solutions of the equation

$$\frac{d^2\gamma_k}{dt^2} + (\omega_k^2 + a\mu_k \cos\omega t)\gamma_k = 0,$$

where

$$\omega_k^2 = \frac{k^2\pi^2}{\rho A L^2}\left(\frac{EIk^2\pi^2}{L^2} - P_o\right)$$

and $\mu_k = \frac{k^2\pi^2}{\rho A L^2}$. In section-4.7 we have seen that using the variable transformation $\tau = \omega t$, $\alpha_k = (\omega_k/\omega)^2$ and $\varepsilon_k = a\mu_k/\omega^2$ this equation can be transformed in to the normalized Mathieu equation

$$\frac{d^2\gamma_k}{d\tau^2} + (\alpha_k + \varepsilon_k \cos\tau)\gamma_k = 0.$$

Furthermore we have also seen in section-4.5.2 that the stability of the origin of the system depends on the parameters $(\alpha_k, \varepsilon_k)$ and that the figure-4.15 shows the regions in the $(\alpha_k, \varepsilon_k)$ space for which the origin is stable and not.

Let us see where the system lies in the parameter space (α, ε) for each mode for a given amplitude of forcing a as the forcing frequency ω varies. We find that

$$\frac{\varepsilon_k}{\alpha_k} = \frac{a}{\left(\frac{EIk^2\pi^2}{L^2} - P_o\right)} = m_k.$$

Thus for a given mode the system lies on the straight-line through the origin in (α, ε) space with slope m_k as ω varies. We also see that As the driving frequency is increased the system moves along this line towards the origin. We also observe that as k increases the line flattens. For most cases it is typical that m_k are very small.

Observe that the figure-4.15 shows that in particular when $\alpha_k \approx 1$ and $\alpha_k \approx 0.25$ the mode is unstable for all k. When $\alpha_k \approx 1$ we see that $\omega \approx \omega_k$ and that this implies the existence of the well known ordinary resonance conditions. In contrast the fact that the modes becomes unstable when $\alpha_k \approx 0.25$ shows that how small the amplitude of forcing may be each mode becomes unstable for forcing frequencies close to 20 times the natural frequency of the mode. This phenomena is called *parametric resonance*.

We also note that when $P_o = P_{cr} \triangleq EI\pi^2/L^2$, $\omega_1 = 0$, and hence $\alpha_1 = 0$. Thus the stability map shows that at this condition for sufficiently small a the first mode is stable. Which implies that the zeroth mode, the straight configuration, is unstable. The consequence of this is that, at this critical load, the slightest perturbation to the system causes the beam to buckle. This phenomena is called *buckling*.

Let us consider the example of a uniform rod of length 1.5 m, 3 cm diameter that is made-up of steel with $E = 208$ GN/m^2, and $\rho = 7780$kg /m^3. If the beam is subject to a tensile load of the form $P(t) = (1000 - 100\cos\omega t)$N, then the first two modes have a natural frequency of 27.4 Hz and 108.7 Hz. For these modes we verify that see that $m_1, m_2 << 1$. The lines are almost flat and thus the chances of exciting the low frequency parametric resonance is very low. For this rod in tension we see that parametric resonance occurs around the very high frequencies of 548 Hz and 2172 Hz.

Let us also consider the case where the axial load is compressive. Let us consider a compressive load that is about 3% of the buckling load $P_{cr} = EI\pi^2/L^2 = 36.3$kN. That is let $P(t) = (1000 - 100\cos\omega t)$. The natural frequency corresponding to the first mode is $\omega_1 \approx 26.7$ Hz. One finds that in this case also $m_1, m_2 << 1$. Thus what one needs to worry about is the parametric resonance that would occur at the high frequency around 534 Hz.

4.7.2 Stabilization of the Inverted Pendulum with Periodic Vertical Forcing

In the previous section we witnessed how high frequency excitations may destabilize an otherwise stable system. In this section we will see how we may achieve the converse. That is, use high frequency excitations to stabilize an unstable equilibrium. The example we consider is the problem of stabilizing the unstable equilibrium of a pendulum.

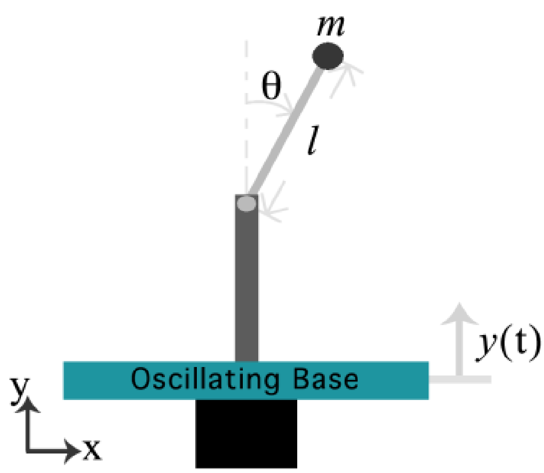


Fig. 4.17 A Base Excited Inverted Pendulum.

If the base of a pendulum is externally exited so that $y(t) = a\cos\omega t$ the equations of motion of the system expressed using an angle θ measured from the upward vertical position is

$$\ddot{\theta} + \left(-\frac{g}{l} + \frac{a\omega^2}{l}\cos\omega t\right)\sin\theta = 0. \tag{4.121}$$

For small displacements from the unstable vertical equilibrium $\sin\theta \approx \theta$ and we have the liberalized system

$$\ddot{\theta} + (\alpha + \varepsilon\cos t)\,\theta = 0. \tag{4.122}$$

where we have made a time scaling by ω and have set $\alpha = -\frac{g}{\omega^2 l}$ and $\varepsilon = \frac{a}{l}$. This is the normalized Mathieu equation we dealt with in the previous section. Since the α are negative we only need to consider the left half side of the stability map shown in Figure-4.15. Figure-4.18 shows a portion of this part of the map.

The question we would like to answer is at what base excitation frequency ω will the vertically unstable equilibrium become stable. From the stability map shown in Figure-4.18 we see that when a/l is small the excitation frequency required for stabilization is very large. For instance if $\varepsilon = a/l = 0.2$ figure-4.18 shows that the equilibrium is stable only if $-\alpha = \frac{g}{\omega^2 l} < 0.02$. Which implies that $\omega > 10\sqrt{\frac{g}{2l}}$. Thus if the pendulum is about 10 cm and the amplitude of base excitation is about 2 cm then one needs an excitation that is greater than 12 Hz.

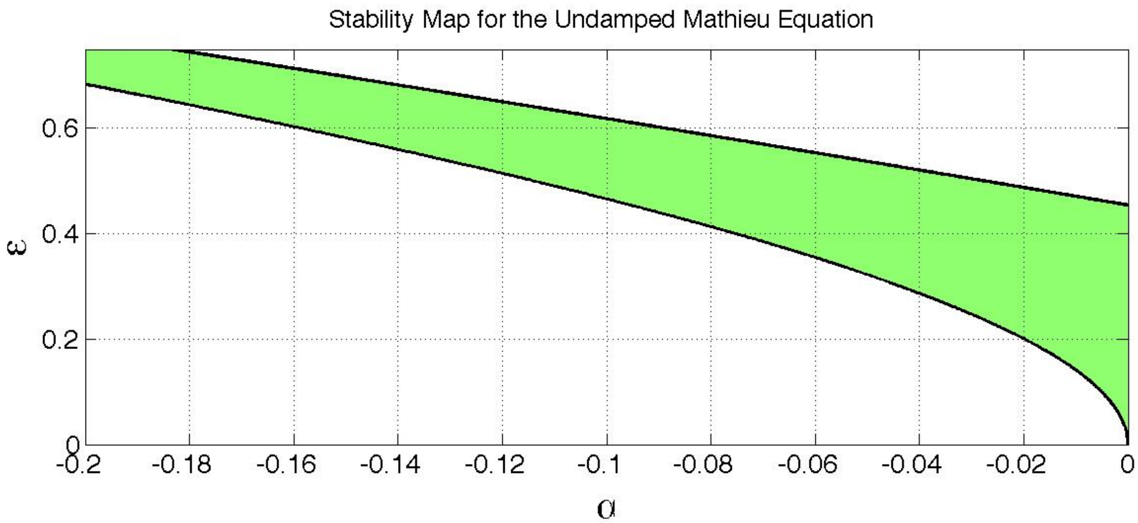


Fig. 4.18 Closeup of the Left Half Side of the Stability Map for the Mathieu Equation. Figures were created by J. M. Berg using a numerical algorithm in the text 'Nonlinear Ordinary Differential Equations' by Jordan and Smith.

4.8 Exercises on Basic Vibration Analysis

Exercise 4.1. Consider the torsional vibration system shown in figure-4.19.

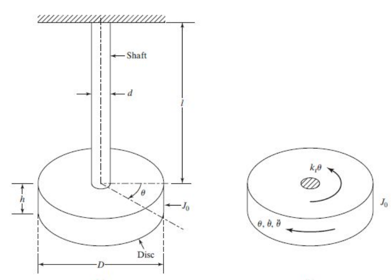


Fig. 4.19 A Tortional Vibration System.

1. Show that for small angular displacements the behavior of θ is given by

$$\ddot{\theta}+2\zeta\omega_n\dot{\theta}+\omega_n^2\theta=0,$$

where $\omega_n^2=K/J_0$, $2\zeta\omega_n=C/J_0$. Here J_0 is the moment of inertia about the axis of rotation of the system, K is the torsional stiffness of the rod, and C is the damping co-efficient of the system.

2. Show that if the damping is low, the response of the system for small initial displacements and velocities is given by

$$\theta(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{\theta}(0)}{\omega_d}\cos(\omega_d t - \pi/2) + \frac{\theta(0)}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi_{IC})\right)1(t).$$

3. Describe the main steps that you would take to experimentally estimate the damping ratio ζ and the frequency of oscillations ω_d of the system.
4. Using the experiment described above estimate the damping ratio and frequency of oscillations of the torsional system in the applied mechanics lab.
5. Using experimental results above estimate the torsional stiffness of the thin rod and calculate the error bounds of your estimate.
6. Validate the accuracy of your results.

Exercise 4.2. Figure 1.42 shows a schematic representation of the Free/Forced vibration apparatus in the applied mechanics lab. The apparatus consists of a beam pivoted at O. A damper is attached to the point A and a spring is attached to the point B. The other end of the spring, P, is constrained to move vertically. When the beam is horizontal and the point P has zero displacement with respect to the reference, that is when $\theta = 0$ and $z = 0$, the spring is at its natural length (unstretched or uncompressed). The spring constant is k the damping constant is C and the moment of inertia of the beam about O is I. Let $OA = L_c$ and $OB = L_k$. Describe a suitable experimental procedure to determine the damping constant of the system. State all assumptions and approximations clearly.

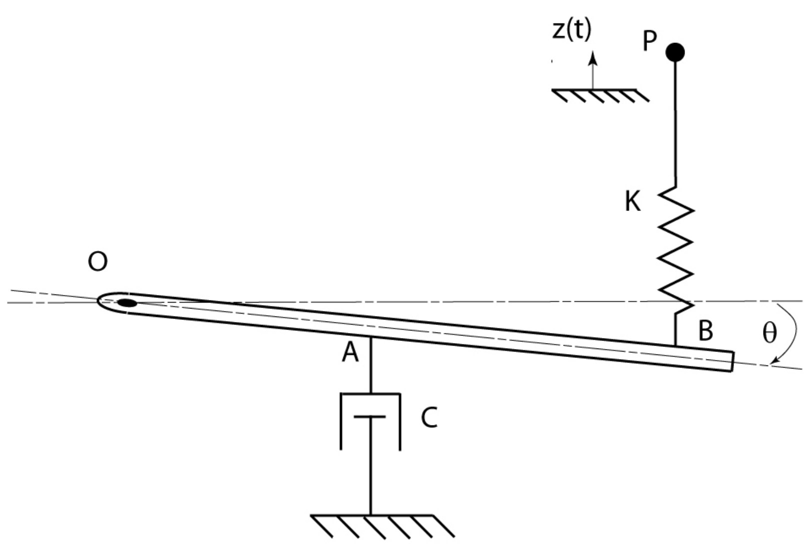


Fig. 4.20 A schematic representation of the Free/Forced Vibration Apparatus in the Applied Mechanics Lab.

Exercise 4.3. Consider the 2-DOF translational system shown in figure-4.21.

4.3.a Find the natural frequencies of the system.
4.3.b Find the normal modes of vibration and the normalized equations.
4.3.c Analytically find the frequency response of the system.
4.3.d Numerically plot the frequency response of the system using the MATLAB for various combinations of parameters of the system. The MATLAB function *bode* in the controls toolbox may be used for this purpose.

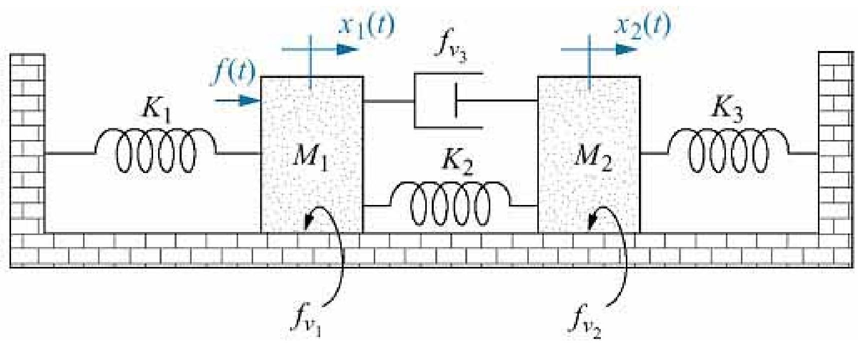


Fig. 4.21 2-DOF Translational Vibratory System

Exercise 4.4. Figure-4.22a shows a schematic of a long rotating shaft mounted on two bearings of damping co-efficient D_1 and D_2 respectively. The shaft is seen to exhibit a torsional vibration. Figure-4.22b shows a simplified approximation of the system. Using this approximation answer the following:

4.4.a Estimate the normal modes of vibration of the shaft.
4.4.b If $T(t) = A\cos(\omega t)$ find the response of θ_2 of the approximated system shown in figure-4.22b as t becomes very large.

Exercise 4.5. An unbalanced small high speed motor is mounted on a heavy platform that is connected to the ground through very stiff springs as shown in figure-4.23. The springs are spaced a distance L away from the center line of the platform and the center of the motor is a distance L_0 away from the center line of the platform. The unbalance distance from the center of rotation is found to be ε and the mass of the rotor of the motor is found to be m. Assuming that the stiffness of the springs in the axial direction is much smaller than the stiffness in the transverse direction. Derive a simplified approximate model for the system and find the normal modes of vibration.

Exercise 4.6. A blower driven by a motor is mounted on the ground using a vibration isolation arrangement similar to that shown in figure-4.24. Due to some modifications in the operation of the system the original blower had to be replaced with a new one with an operating region between 1000 r.p.m and 2000 r.p.m. The new motor when operated at 990 r. p. m. and 1710 r. p. m. exhibited violent vibrations. You are required to design a vibration absorber to mitigate the vibrations. As a first step one would typically analyzes the approximate behavior of the system and attempt to theoretically predict its behavior.

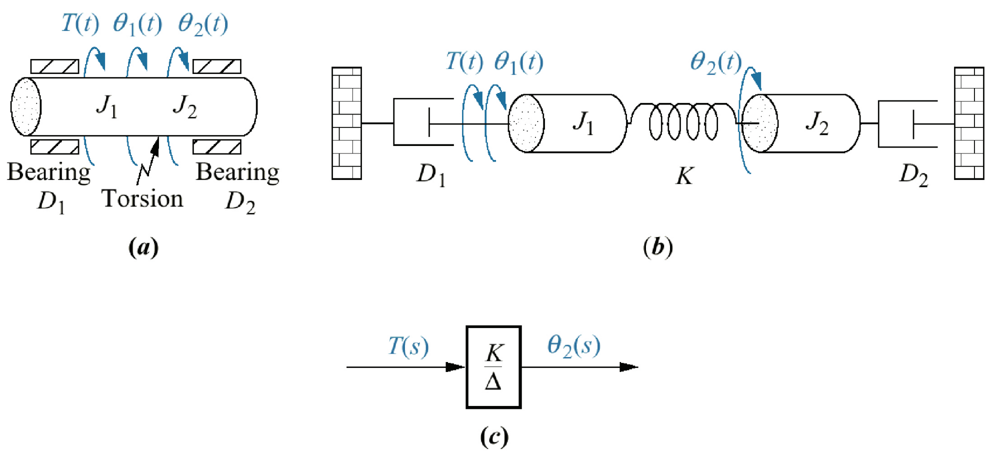


Fig. 4.22 Torsional vibration of a long rotating shaft.

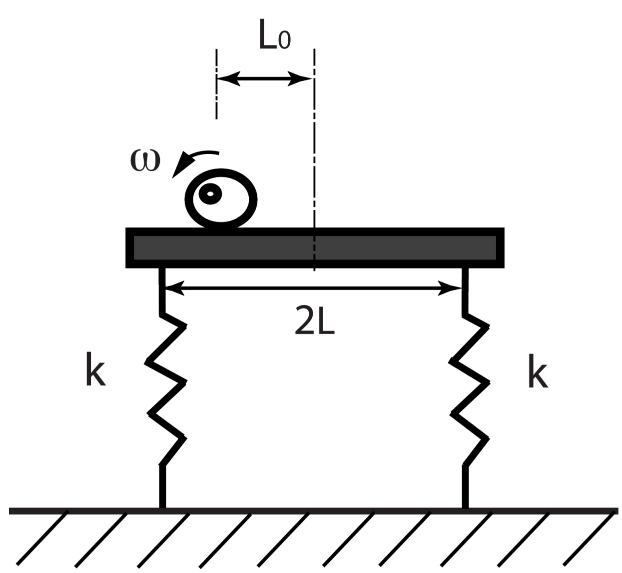


Fig. 4.23 An Unbalance motor on a platform.

1. Explain the cause of vibration in the system.
2. Assuming that the motor mass and eccentricities are negligible compared to that of the blower, the system can be assumed to behave similarly to that of the coupled vibration setup in the Applied Mechanics lab. For small deflections and velocities the experimental setup in the lab can be approximately modeled by a lumped parameter system shown in figure-4.25 where the structural damping in the springs is assumed to be negligible. Show that for small displacements, and angles the equations of motion of the system are given by

$$M\ddot{z} + Kz = F(t),$$

where

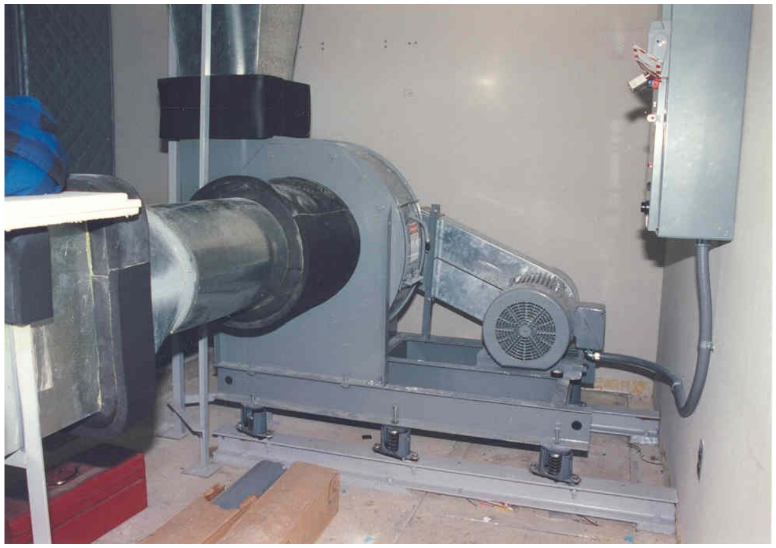


Fig. 4.24 A Vibration Isolation Mounting of Motor Driving a Blower.

$$z(t)=\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix},\quad F(t)=\begin{bmatrix} f(t) \\ c\,f(t) \end{bmatrix},\quad M=\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix},K=\begin{bmatrix} 2k & -k(l_2-l_1) \\ -k(l_2-l_1) & k(l_1^2+l_2^2) \end{bmatrix}.$$

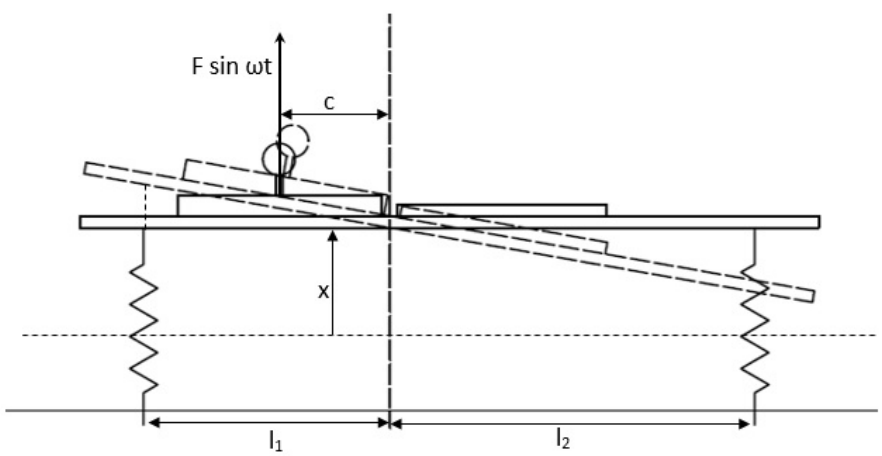


Fig. 4.25 A 2-DOF coupled torsional plus translational system

3. Write down these equations in the form

$$\ddot{z}+\Omega z = M^{-1}F(t),$$

where

$$\Omega = \begin{bmatrix} \omega_{n1}^2 & -(l_1^2+l_2^2)\mu\omega_{n1}^2 \\ -2\mu\alpha^2\omega_{n1}^2 & \alpha^2\omega_{n1}^2 \end{bmatrix}.$$

with

$$\omega_{n1}^2 = \frac{2k}{m},\ \ \omega_{n2}^2 = \frac{k(l_1^2+l_2^2)}{I},\ \ \mu = \frac{(l_2-l_1)}{2(l_1^2+l_2^2)},\ \ \alpha = \frac{\omega_{n2}}{\omega_{n1}} = \sqrt{\frac{m(l_1^2+l_2^2)}{2I}},$$

4. State the conditions for the existence of synchronized motion of the system of the form

$$z_m(t) = \bar{z}_m\cos(\omega t+\phi),$$

for some constant 2×2 matrix $\bar{z}_m = [\bar{x}_m \ \ \bar{\theta}_m]^T$.

5. Show that there exists two distinct types of synchronized solutions of the form

$$\begin{aligned} z_{m1}(t) &= \bar{z}_{m1}\cos(\omega_1 t+\phi_1), \\ z_{m2}(t) &= \bar{z}_{m2}\cos(\omega_2 t+\phi_2), \end{aligned}$$

where

$$\omega_1 = \omega_{n1}\left(\frac{(1+\alpha^2)-\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}}{2}\right)^{\frac{1}{2}},$$

$$\omega_2 = \omega_{n1}\left(\frac{(1+\alpha^2)+\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}}{2}\right)^{\frac{1}{2}},$$

and $\bar{z}_{m1} = [\bar{x}_{m1} \ \ \bar{\theta}_{m2}]^T$ and $\bar{z}_{m2} = [\bar{x}_{m2} \ \ \bar{\theta}_{m2}]^T$ are the eigenvectors corresponding to the eigenvalues ω_1^2 and ω_2^2 respectively while ϕ_1 and ϕ_2 are arbitrary constants.

6. Show that

$$\frac{\bar{x}_{m1}}{\bar{\theta}_{m1}} = \frac{\alpha^2-1+\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}}{4\mu\alpha^2},$$

$$\frac{\bar{x}_{m2}}{\bar{\theta}_{m2}} = \frac{\alpha^2-1-\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}}{4\mu\alpha^2}.$$

or alternatively

$$\frac{\bar{x}_{m1}}{\bar{\theta}_{m1}} = \frac{2(l_1^2+l_2^2)\mu}{1-\alpha^2+\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}},$$

$$\frac{\bar{x}_{m2}}{\bar{\theta}_{m2}} = \frac{2(l_1^2+l_2^2)\mu}{1-\alpha^2-\sqrt{(1+\alpha^2)^2-4\alpha^2(1-2\mu^2(l_1^2+l_2^2))}}.$$

7. Describe the qualitative behavior of the two modes of vibration.
8. Discuss the case if the blower is mounted very close to the middle of the platform.
9. The mass of the motor is fairly accurately known. However due to the use over time the spring constants of the mounts are poorly known. Describe a method that you would use to estimate the spring constant of the mount.
10. Describe how you would validate the estimate of the spring constants.
11. Estimate the spring constant of the coupled 2DOF vibratory system in the lab and discuss the validity of your estimate.

Exercise 4.7. A motor that weighs 200 kg is mounted on the floor using a stiff platform of overall stiffness estimated to be 80 kN/m. The unbalances in the rotor induces a periodic forcing on the platform. The operating conditions of the motor are between 60 r.p.m. and 300 r.p.m. Design a tuned mass damper for the system. State clearly all assumptions and approximations made.

Exercise 4.8. Consider the 3-DOF translational vibratory system shown in figure-4.26. If $k_1 = k_2 = k_3 = k$ and $C_1 = C_2 = C_3 = C$ find the initial condition response of the system when only the first mass is given an initial displacement of x_{10}.

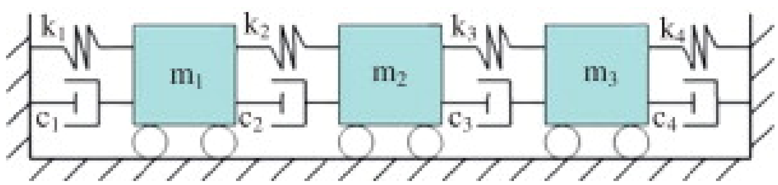


Fig. 4.26 A 3-DOF translational vibratory system

Exercise 4.9. Find the modes of vibration of the 3-DOF translational vibratory system shown in figure-4.27. Comment on the form of the each mode of vibration.

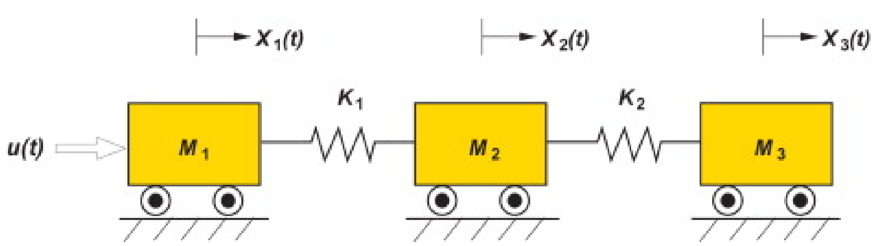


Fig. 4.27 A 3-DOF translational vibratory system

Exercise 4.10. Find the resonance frequencies of the 3-DOF translational vibratory system shown in figure-4.28. Assuming zero initial conditions find the response of the system when $z(t) = 1(t)$ the unit step function.

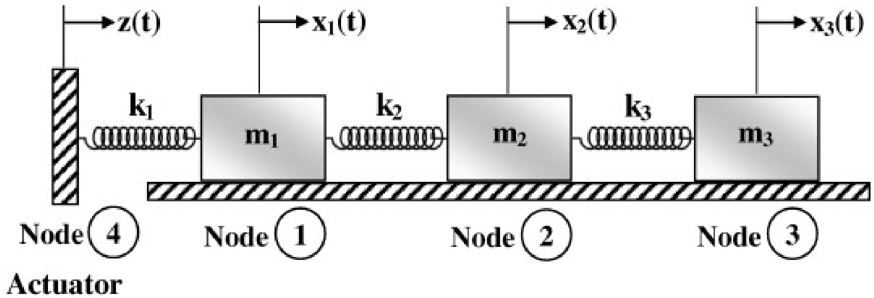


Fig. 4.28 A 3-DOF translational vibratory system

Exercise 4.11. A uniform shaft of 9.52 mm diameter is supported in ball bearings and carries four discs A, B, C, D. The bearings are placed close to the discs, so that the sag of the shaft is reduced to a minimum. The wheel A carries a small spindle which is supported in ball bearings. The spindle carries an out-of balance mass and is belt driven by a variable speed DC motor. A schematic of the setup is shown in figure-4.29. Answer the following with detailed justification:

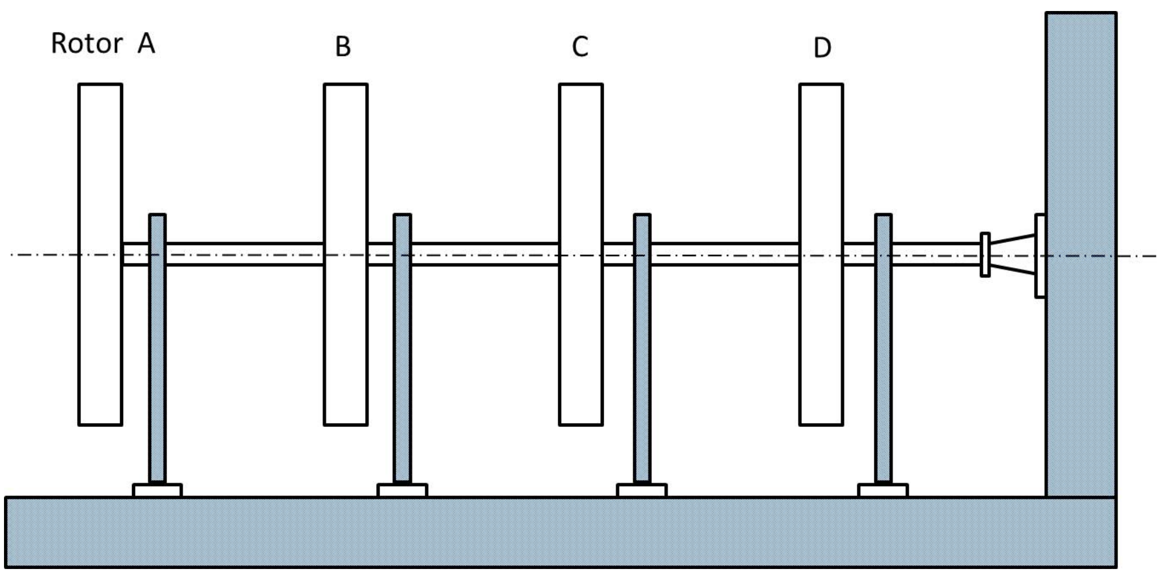


Fig. 4.29 A Multi-Rotor Vibratory System

1. Assuming that the damping in the system is negligible, show that for small angular displacements the equations of motion of the system are given by

$$I\ddot{\theta} + K\theta = T(t),$$

where

$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \\ \theta_4(t) \end{bmatrix}, \quad T(t) = \begin{bmatrix} f(t) \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$I = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_4 \end{bmatrix}, K = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & (k_1+k_2) & -k_2 & 0 \\ 0 & -k_2 & (k_2+k_3) & -k_3 \\ 0 & 0 & -k_3 & (k_3+k_4) \end{bmatrix}.$$

2. Write down these equations in the form

$$\ddot{\theta} + \Omega\theta = M^{-1}F(t),$$

where

$$\Omega = \begin{bmatrix} \omega_{n1}^2 & -\omega_{n1}^2 & 0 & 0 \\ -\mu_1\omega_{n1}^2 & (\mu_1\omega_{n1}^2+\omega_{n2}^2) & -\omega_{n2}^2 & 0 \\ 0 & -\mu_2\omega_{n2}^2 & (\mu_2\omega_{n2}^2+\omega_{n3}^2) & -\omega_{n3}^2 \\ 0 & 0 & -\mu_3\omega_{n3}^2 & (\mu_3\omega_{n3}^2+\omega_{n4}^2) \end{bmatrix}.$$

with

$$\omega_{n1}^2 = \frac{k_1}{I_1}, \ \ \omega_{n2}^2 = \frac{k_2}{I_2}, \ \ \omega_{n3}^2 = \frac{k_3}{I_3}, \ \ \omega_{n4}^2 = \frac{k_4}{I_4},$$

$$\mu_1 = \frac{I_1}{I_2}, \ \ \mu_2 = \frac{I_2}{I_3}, \ \ \mu_3 = \frac{I_3}{I_4},$$

3. For the unforced undamped system, state the conditions for the existence of synchronized motion of the form

$$\theta_m(t) = \bar{z}_m \cos(\omega t + \phi),$$

for some constant 4×1 matrix $\bar{\theta}_m = [\bar{\theta}_{1_m} \quad \bar{\theta}_{2_m} \quad \bar{\theta}_{3_m} \quad \bar{\theta}_{4_m}]^T$.
4. Find the modes of vibration of the system.
5. Qualitatively describe the behavior of each of the modes of vibration.

Exercise 4.12. Consider the cantilevered beam shown in figure-4.30. The deflection of the beam is modeled by the Euler's beam equaion

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4} = q(x,t)$$

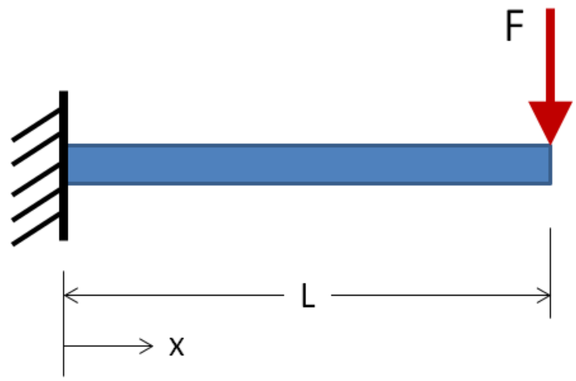


Fig. 4.30 A Cantilevered Beam

Find the modes of vibration of the cantilevered beam and sketch the first three modes of vibration.

Exercise 4.13.
The transverse deflection of unforced rotating shafts can be approximately modeled by

$$\frac{\partial^2 y}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4}-\frac{EI}{\kappa GA}\frac{\partial^4 y}{\partial t^2 \partial x^2}=0.$$

The shaft is simply supported at the ends and hence satisfy the boundary conditions

$$\begin{aligned} y(0,t)&=0, \quad y(L,t)=0,\\ \frac{\partial^2 y}{\partial x^2}(0,t)&=0, \quad \frac{\partial^2 y}{\partial x^2}(L,t)=0, \end{aligned}$$

Find the modes of vibration and the general solution of the system.

Exercise 4.14. Consider the vibration of a thin circular membrane such as in the case of a drum. The deflection of the membrane of such a rotationally symmetric system is most conveniently described using polar coordinates (r,θ). That is consider the deflection of the membrane to be a function of (t,r,θ). Expressing the wave equation in polar coordinates, one finds that the deflection of the membrane satisfies

$$\frac{\partial^2 y}{\partial t^2}-c^2\left(\frac{\partial^2 y}{\partial r^2}+\frac{1}{r}\frac{\partial y}{\partial r}+\frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2}\right)=0,$$

and the boundary conditions $u(t,r_0,\theta)=0$. Find the modes of vibrations.